In initially we note that by Conservation of Energy

$$\text{Total Energy}(E) = \text{Potential Energy}(P) + \text{Kinetic Energy}(K)$$

where Potential Energy is given by $P = mg(\text{height})$ with $m =$ mass and $g =$ acceleration due to gravity, and Kinetic Energy is given by $K = \frac{1}{2}mv^2$ with $m =$ mass and $v =$ velocity.

We are interested in letting an object fall under the influence of gravity from $(0,0)$ to $(\pi, 2)$ along a path $c(t) = (x(t), y(t))$. In our case,

$$E = mg(2-y) + \frac{1}{2}mv^2$$

with initial data

$$P_0 = 2mg \quad \text{and} \quad K_0 = 0 \quad \text{so} \quad E = E_0 = P_0 + K_0 = 2mg.$$

In other words, the total energy in our system is $2mg$; this will be important below. It is worth noting that in our system y measures the distance an object has fallen since $y = 0$ is at the 'top' and $y = 2$ is at the 'bottom'.

After an object has fallen y (pick you favorite units, as long as g corresponds, meters, feet, parsecs, etc.) without friction under uniform gravity we have

$$2mg = (2-y)mg + \frac{1}{2}mv^2$$

so that

$$mgy = \frac{1}{2}mv^2$$

and finally, solving for v gives

$$v = \sqrt{2gy}. \tag{1}$$

Since (Rate)(Time) = Distance, or Time = Distance/Rate, we play the standard 172 game and chop things up into little pieces and approximate giving

$$\Delta T = \frac{\Delta\text{Distance}}{\text{Rate}} = \frac{\Delta s}{v}$$

where s is the arc length and v is the velocity, i.e. $v = \sqrt{2gy}$. So, the the time to reach the 'bottom' along a curve $c(t) = (x(t), y(t))$ with $c(a) = (0,0)$ and $c(b) = (\pi, 2)$ is approximated by

$$\text{Time} = \sum \Delta T = \sum \frac{\Delta s}{v} = \sum \frac{\Delta s}{\sqrt{2gy}}.$$

Taking the limit gives

$$\text{Time} = \int_a^b \sqrt{\frac{(x'(t))^2 + (y'(t))^2}{2gy(t)}}\, dt. \tag{2}$$

Earlier we saw that for the following curves connecting $(0,0)$ to $(\pi, 2)$, we had the following arc lengths.

- The line segment: $x = \pi t$, $y = 2t$, for $t \in [0,1]$, $s = \sqrt{\pi^2 + 4} \approx 3.724$.
- The parabola: $x = \pi t^2$, $y = 2t$, for $t \in [0,1]$, $s = \frac{1}{\pi}\left(\pi\sqrt{1+\pi^2} + \ln\left|\sqrt{1+\pi^2} + \pi\right|\right) \approx 3.890$.
- The ellipse: $x = \pi(1 - \cos t)$, $y = 2\sin t$, for $t \in [0, \frac{\pi}{2}]$, $s \approx 4.088$.
- The cycloid: $x = t - \sin t$, $y = 1 - \cos t$, for $t \in [0, \pi]$, $s = 4$.

Question: What is the path of fastest descent? Is it the shortest path, i.e. the line? Is it one of the others? This is a classic question, the **Brachistochrone problem**, posed by Johann Bernoulli in June 1696.

Answer: By May 1697, five (possibly six[1]) mathematicians responded with solutions to the **Brachistochrone problem** - the cycloid is the path of fastest descent. They were Johann Bernoulli himself, his brother Jakob Bernoulli, Guillaume de l'Hôpital (of l'Hôpital's Rule), Gottfried Leibniz, and Issac Newton. This is a remarkably classic question originally answered by many of the big names of calculus.

We don't have the machinery available to provide a complete answer; however, we can compute the time required for the four curves we are considering using (2), assuming measurements in meters and $g = 9.8\frac{m}{s^2}$.

- Line: $T = \int_0^1 \sqrt{\frac{\pi^2+4}{2g2t}}\,dt = \sqrt{\frac{\pi^2+4}{g}} \approx 1.190s.$
- Parabola: $T = \int_0^1 \sqrt{\frac{4\pi^2t^2+4}{2g2t}}\,dt \approx 1.013s.$
- Ellipse: $T = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi^2\sin^2 t + 4\cos^2 t}{2g2\cos t}}\,dt \approx 1.166s.$
- Cycloid: $T = \int_0^{\pi} \sqrt{\frac{(1-\cos t)^2+\sin^2 t}{2g(1-\cos t)}}\,dt = \int_0^{\pi} \sqrt{\frac{2-2\cos t}{2g(1-\cos t)}}\,dt = \frac{1}{\sqrt{g}}\int_0^{\pi} dt = \frac{\pi}{\sqrt{g}} \approx 1.004s.$

Of the considered curves, the cycloid has the shortest time. Using more advanced methods, we can show that the cycloid is the path of fastest descent, i.e. the solution to the Brachistochrone problem.

A good mathematician realizes when a problem cannot be solved, and changes the problem to one that can be solved. With that in mind, we consider a related and equally classic problem.

The cycloid also solves the **Tautochrone problem**, i.e. an object following the curve descends to the bottom in the same amount of time independent of its starting position on the curve - originally solved by Christiaan Huygens in 1659.

If we start at $0 \le t_0 \le \pi$ on the cycloid, we are at the point $(t_0 - \sin t_0, 1 - \cos t_0)$. In that case the distance fallen changes from $y = (1-\cos t)$ to $(1-\cos t) - (1-\cos t_0) = (\cos t_0 - \cos t)$. Replacing $y(t)$ with $(\cos t_0 - \cos t)$ in (1) and therefore (2) gives

$$\text{Time} = \int_{t_0}^{\pi} \sqrt{\frac{(1-\cos t)^2+\sin^2 t}{2g(\cos t_0 - \cos t)}}\,dt = \frac{1}{\sqrt{g}}\int_{t_0}^{\pi} \sqrt{\frac{1-\cos t}{\cos t_0 - \cos t}}\,dt.$$

Now we use the following trig identities: $1-\cos t = 2\sin^2(t/2)$ and $\cos t_0 - \cos t = (1+\cos t_0)-(1+\cos t) = 2\cos^2(t_0/2) - 2\cos^2(t/2)$ to rewrite the integral as

$$\text{Time} = \frac{1}{\sqrt{g}}\int_{t_0}^{\pi} \sqrt{\frac{2\sin^2(t/2)}{2\cos^2(t_0/2) - 2\cos^2(t/2)}}\,dt = \frac{1}{\sqrt{g}}\int_{t_0}^{\pi} \frac{\sin(t/2)}{\cos(t_0/2)\sqrt{1-\cos^2(t/2)/\cos^2(t_0/2)}}\,dt.$$

Finally, we note that t_0 and $\cos(t_0/2)$ are both constant and then make the substitution

$$u = \frac{\cos(t/2)}{\cos(t_0/2)}, \text{ so } du = \frac{(-1/2)\sin(t/2)}{\cos(t_0/2)}, \text{ with limits } t = t_0 \to u = 1, t = \pi \to u = 0$$

giving

$$\text{Time} = \frac{-2}{\sqrt{g}}\int_1^0 \frac{du}{\sqrt{1-u^2}} = \frac{-2}{\sqrt{g}}\arcsin u\Big|_1^0 = \frac{\pi}{\sqrt{g}}.$$

[1] Wikipedia also lists Ehrenfried Walther von Tschirhaus as a mathematician with a solution, but the source I have - *Journey Through Genius* by William Dunham - only lists five.