Math 172 Final Exam Review

1. A. $\displaystyle\int (x+3)\sin 2x\,dx = -\frac{(x+3)\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx = \frac{\sin 2x}{4} - \frac{(x+3)\cos 2x}{2} + C$

$u = x+3 \qquad dv = \sin 2x\,dx$

$du = dx \qquad v = -\dfrac{\cos 2x}{2}$

B. $\displaystyle\int \sin x \cos^7 x\,dx = -\int u^7\,du = -\frac{\cos^8 x}{8} + C$

$u = \cos x$

$du = -\sin x\,dx$

$\sin 2\theta = 2\sin\theta\cos\theta$

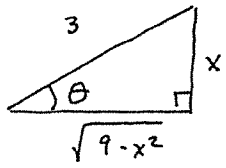
C. $\displaystyle\int \sqrt{9-x^2}\,dx = \int 9\cos^2\theta\,d\theta = \frac{9}{2}\int (1+\cos 2\theta)\,d\theta = \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

let $x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

So $9 - x^2 = 9 - 9\sin^2\theta$

$= 9\cos^2\theta$



$\displaystyle = \frac{9}{2}\left(\theta + \sin\theta\cos\theta + C\right)$

$\displaystyle = \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{9}\right) + C$

D. $\displaystyle\int \frac{7x-6}{(2x+1)(x+3)}\,dx = \int\left(\frac{1}{2x+1} + \frac{3}{x+3}\right)dx = \frac{1}{2}\ln|2x+1| + 3\ln|x+3| + C$

$$\frac{7x+6}{(2x+1)(x+3)} = \frac{A}{2x+1} + \frac{B}{x+3}$$

$$7x+6 = A(x+3) + B(2x+1)$$

let $x = -3$: $\quad -15 = B(-5) \Rightarrow B = 3$

$x = -\frac{1}{2}$: $\quad \frac{5}{2} = A\left(\frac{5}{2}\right) \Rightarrow A = 1$

E. $\displaystyle\int x^2 e^{-4x}\,dx = -\frac{1}{4}x^2e^{-4x} + \int \frac{1}{2}xe^{-4x}\,dx = -\frac{1}{4}x^2e^{-4x} - \frac{1}{8}xe^{-4x} + \int \frac{1}{8}e^{-4x}\,dx$

$u = x^2 \qquad dv = e^{-4x}\,dx$

$du = 2x\,dx \qquad v = -\frac{1}{4}e^{-4x}$

$u = \frac{1}{2}x \qquad dv = e^{-4x}\,dx$

$du = \frac{1}{2}dx \qquad v = -\frac{1}{4}e^{-4x}$

$$= -\frac{1}{4}x^2e^{-4x} - \frac{1}{8}xe^{-4x} - \frac{1}{32}e^{-4x} + C$$

F. $\displaystyle\int \sec^4 4\theta \tan^4 4\theta\,d\theta = \int \underbrace{(\tan^2 4\theta + 1)}_{=\sec^2 4\theta} \tan^4 4\theta \sec^2 4\theta\,d\theta = \frac{1}{4}\int (u^2+1)\,u^4\,du$

let $u = \tan 4\theta$

$du = 4\sec^2 4\theta\,d\theta$

$$= \frac{1}{4}\left(\frac{\tan^7 4\theta}{7} + \frac{\tan^5 4\theta}{5}\right) + C$$

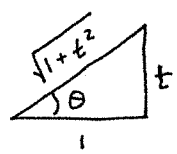
G. $\displaystyle\int \frac{dt}{t^4+2t^2+1} = \int \frac{dt}{(t^2+1)^2} = \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$

Let $t = \tan\theta$

$dt = \sec^2\theta\,d\theta$

so $t^2+1 = \tan^2\theta + 1$

$= \sec^2\theta$



$\sin 2\theta = 2\cos\theta\sin\theta$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{2}\left(\arctan t + \frac{(t)(1)}{(\sqrt{1+t^2})^2}\right) + C$$

$$= \frac{1}{2}\left(\arctan t + \frac{t}{1+t^2}\right) + C$$

H. $\displaystyle\int \frac{4x^2+3x+2}{x^3+x^2}\,dx = \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1}\right)dx$

$$= \ln|x| - \frac{2}{x} + 3\ln|x+1| + C$$

$$\frac{4x^2+3x+2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$4x^2+3x+2 = Ax(x+1) + B(x+1) + Cx^2$$

Let $x=0$: $\quad 2 = B$

Let $x=-1$: $\quad 3 = C$

Equate coeff to find A

x^2: $\quad 4 = A + C \Rightarrow A = 1$

I. $\displaystyle\int_1^e t \ln t \, dt = \frac{1}{2}t^2 \ln t \Big|_1^e - \int_1^e \frac{1}{2} t \, dt = \frac{1}{2}e^2 - \frac{1}{4}t^2 \Big|_1^e = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2+1)$

$u = \ln t \qquad dv = t\,dt$

$du = \frac{1}{t}dt \qquad v = \frac{1}{2}t^2$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ $\qquad$ $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

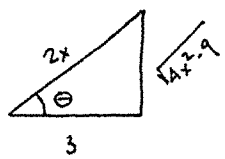
J. $\displaystyle\int \sin^2(\pi x)\cos^2(\pi x)\,dx = \frac{1}{4}\int \sin^2(2\pi x)\,dx = \frac{1}{8}\int \left(1-\cos(4\pi x)\right)dx$

$\displaystyle = \frac{1}{8}\left(x - \frac{1}{4\pi}\sin(4\pi x)\right) + C$

K. $\displaystyle\int \frac{\sqrt{4x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{\frac{3}{2}\sec\theta}\cdot\frac{3}{2}\sec\theta\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)\,d\theta$

Let $2x = 3\sec\theta$

$2dx = 3\sec\theta\tan\theta\,d\theta$



$= 3(\tan\theta - \theta) + C$

So $4x^2 - 9 = 9\sec^2\theta - 9$

$= 9\tan^2\theta$

$\displaystyle = 3\left(\frac{\sqrt{4x^2-9}}{3} - \operatorname{arcsec}\left(\frac{2x}{3}\right)\right) + C$

L. $\displaystyle\int \frac{5x+6}{(x-2)(x^2+4)}\,dx = \int\left(\frac{2}{x-2} - \frac{2x}{x^2+4} + \frac{1}{x^2+2^2}\right)dx$

$u = x^2+4$

$du = 2x\,dx$

$\displaystyle\frac{5x+6}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$

$= 2\ln|x-2| - \ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

$5x+6 = A(x^2+4) + (Bx+C)(x-2)$

Let $x = 2$: $\quad 16 = A(8) \Rightarrow A = 2$

Equat coeff: $\underline{x^2}$ $\qquad$ $\underline{x^0}$

$0 = A + B \qquad 6 = 4A - 2C$

So $B = -2 \qquad$ So $C = 1$

Note: $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

is given information.

M. $\int (x+\sin x)^2\,dx = \int (x^2 + 2x\sin x + \sin^2 x)\,dx = \frac{x^3}{3} + 2\sin x - 2x\cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

<u>Note:</u>

$$2\int x\sin x\,dx = -2x\cos x + 2\int \cos x\,dx = -2x\cos x + 2\sin x + C$$

$u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$u = x^2+4x+13$
$du = (2x+4)\,dx$

N. $\int \frac{x^2}{x^2+4x+13}\,dx = \int \left(\frac{x^2+4x+13}{x^2+4x+13} - \frac{4x+13}{x^2+4x+13}\right)dx = \int \left(1 - \frac{2(2x+4)}{x^2+4x+13} - \frac{5}{(x+2)^2+3^2}\right)dx$

$$= x - 2\ln(x^2+4x+13) - \frac{5}{3}\arctan\left(\frac{x+2}{3}\right) + C$$

O. $\int \sin 2\theta\, e^{\sin\theta}\,d\theta = \int 2\sin\theta\cos\theta\, e^{\sin\theta}\,d\theta = \int 2x e^x\,dx = 2xe^x - \int 2e^x\,dx$

Let $x = \sin\theta$ $\qquad u = 2x \qquad dv = e^x dx$

$dx = \cos\theta\,d\theta \qquad du = 2dx \qquad v = e^x \qquad = 2xe^x - 2e^x + C$

$$= 2e^{\sin\theta}(\sin\theta - 1) + C$$

P. $\int \sec^2 x \sec^3(\tan x)\tan^3(\tan x)\,dx = \int \sec^3\theta \tan^3\theta\,d\theta = \int \sec^2\theta\,(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$

$\tan^2\theta$

Let $\theta = \tan x$ $\qquad u = \sec\theta \qquad du = \sec\theta\tan\theta\,d\theta$

$d\theta = \sec^2 x\,dx$

$$= \int (u^4 - u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\sec^5(\tan x) - \frac{1}{3}\sec^3(\tan x) + C$$

Q. $\displaystyle\int \frac{e^{2x}}{e^{2x}+5e^{x}+6}\,dx = \int \frac{u}{u^2+5u+6}\,du = \int\left(\frac{3}{u+3} - \frac{2}{u+2}\right)du$

Let $u = e^x$

$du = e^x\,dx$

$\dfrac{u}{(u+3)(u+2)} = \dfrac{A}{u+3} + \dfrac{B}{u+2}$

$= 3\ln(e^x+3) - 2\ln(e^x+2) + C$

$u = A(u+2) + B(u+3)$

let $u=-2$: $-2 = B$

$u=-3$: $-3 = -A \Rightarrow A = 3$

R. $\displaystyle\int_0^{\pi/2} \cos x\sqrt{1+3\sin^2 x}\,dx = \frac{1}{\sqrt{3}}\int_0^{\pi/3} \sec^3\theta\,d\theta = \frac{1}{2\sqrt{3}}\Big(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big)\Big|_0^{\pi/3}$

Let $\tan\theta = \sqrt{3}\sin x$

$\sec^2\theta\,d\theta = \sqrt{3}\cos x\,dx$

$= \dfrac{1}{2\sqrt{3}}\left(2\cdot\sqrt{3} + \ln|2+\sqrt{3}|\right)$

x	$\longmapsto$	θ
0		0
$\pi/2$		$\pi/3$

2. A $x = t+1$, $y = t^3+1$ so $x-1 = t$ so $y = (x-1)^3+1$

B. $x = e^{2t}+1$, $y = e^{6t}+1$ so $x-1 = e^{2t}$ so $y = (x-1)^3+1$

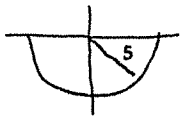
C. $x = \dfrac{2+t}{1+t} = 1 + \dfrac{1}{1+t}$

$y = \dfrac{1}{(1+t)^3}+1$ so $x-1 = \dfrac{1}{1+t}$ so $y = (x-1)^3+1$

D. $x = 5\cos t$

$y = 5\sin t$

$t \in [\pi, 2\pi]$



so $y = -\sqrt{25-x^2}$

3. A. $x = 4t+1$ $\quad x' = 4$

$y = 3t-1$ $\quad y' = 3$

$0 \le t \le 2$

so $ds = \sqrt{(x')^2+(y')^2}\,dt = 5\,dt$

$$S = \int_0^2 5\,dt = 10$$

B. $x = 3t^2+3$ $\quad x' = 6t$ $\quad (x')^2+(y')^2 = 36t^2+9t^4$

$y = t^3-2$ $\quad y' = 3t^2$ $\quad ds = \sqrt{36t^2+9t^4}\,dt$

$0 \le t \le \sqrt{5}$ $\quad = 3t\sqrt{4+t^2}\,dt$

$$S = \int_0^{\sqrt{5}} 3t\sqrt{4+t^2}\,dt = \frac{3}{2}\int_4^9 u^{1/2}\,du$$

$u = 4+t^2$

$du = 2t\,dt$

t	$\longmapsto$	u
0		4
$\sqrt{5}$		9

$$= u^{3/2}\Big|_4^9 = 27-8$$

$$= 19$$

C. $x = e^{2t}\cos t$

$y = e^{2t}\sin t$ $\quad$ <u>see 7.E.</u>

$t \in [0, \pi]$

This is a <u>cycloid</u>.

D. $x = t - \sin t$ $\quad x' = 1-\cos t$ $\quad (x')^2+(y')^2 = 1-2\cos t+\cos^2 t+\sin^2 t$

$y = 1-\cos t$ $\quad y' = \sin t$ $\quad = 2-2\cos t$

$t \in [0, 2\pi]$ $\quad = 2(1-\cos t) \longleftarrow 1-\cos t = 2\sin^2(t/2)$

$= 4\sin^2 t/2$ $\quad$ so $ds = 2\sin t/2\,dt$

$$S = \int_0^{2\pi} ds = \int_0^{2\pi} 2\sin t/2\,dt = -4\cos(t/2)\Big|_0^{2\pi} = -4(-1-1) = 8$$

4. A. $x = 2\sin t$ $\quad x' = 2\cos t$ $\quad x'(\pi/6) = \sqrt{3}$

$y = t$ $\quad y' = 1$ $\quad y'(\pi/6) = 1$

so $\left.\dfrac{dy}{dx}\right|_{t=\pi/6} = \dfrac{1}{\sqrt{3}}$ & $\left.\dfrac{ds}{dt}\right|_{t=\pi/6} = \sqrt{(\sqrt{3})^2+1^2} = 2$

B. $x = \dfrac{t+2}{4}$ $\quad x' = \dfrac{1}{4}$

$y = \dfrac{t-3}{3}$ $\quad y' = \dfrac{1}{3}$

so $\left.\dfrac{dy}{dx}\right|_{t=\frac{-\pi^3}{e^{37}}} = \dfrac{1/3}{1/4} = 4/3$ & $\left.\dfrac{ds}{dt}\right|_{t=\text{ugly number}} = \sqrt{\dfrac{1}{16}+\dfrac{1}{9}} = \dfrac{5}{12}$

4.c. $x = 2\cos 2t$ $\quad x' = -4\sin 2t$ $\quad x'\left(\frac{3\pi}{8}\right) = -4\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y = \sin 2t$ $\quad y' = 2\cos 2t$ $\quad y'\left(\frac{3\pi}{8}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

so $\left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{8}} = \frac{-\sqrt{2}}{-2\sqrt{2}} = 1/2$ $\qquad \left.\frac{ds}{dt}\right|_{t=\frac{3\pi}{8}} = \sqrt{(-2\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{10}$

D. $x = \frac{t}{t+1}$ $\quad x' = \frac{1}{(t+1)^2}$ $\quad x'(1) = \frac{1}{4}$

$y = t^2$ $\quad y' = 2t$ $\quad y'(1) = 2$

so $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{1/4} = 8$ $\qquad \left.\frac{ds}{dt}\right|_{t=1} = \sqrt{\frac{1}{16} + 4} = \frac{\sqrt{65}}{4}$

5. A. $r = 2$ so $x^2 + y^2 = 4$

B. $r = \frac{3}{\cos\theta}$ so $r\cos\theta = 3$ OR $x = 3$

C. $r = 2\csc\theta$ so $r\sin\theta = 2$ OR $y = 2$

D. $r = \frac{6}{2\cos\theta - 3\sin\theta}$ so $2r\cos\theta - 3r\sin\theta = 6$ OR $2x - 3y = 6$

E. $r = -2\sin\theta$ so $r^2 = -2r\sin\theta$ OR $x^2 + y^2 = -2y$

OR if you want to be fancy,

$x^2 + (y+1)^2 = 1$

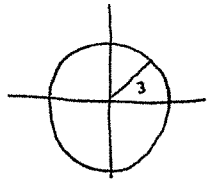
F. $\theta = \pi/4$ OR $y = x$

G. $y = x^2$ so $r\sin\theta = r^2\cos^2\theta$ so $r = \frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta$

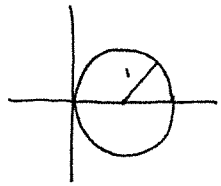
H. $y = 3x + 2$ so $r\sin\theta - 3r\cos\theta = 2$ so $r = \frac{2}{\sin\theta - 3\cos\theta}$

I. $(x-2)^2 + y^2 = 4$ so $x^2 - 4x + y^2 = 0$ OR $r^2 = 4r\cos\theta$ so $r = 4\cos\theta$

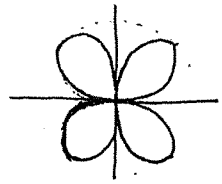
6. A. $r = 3$



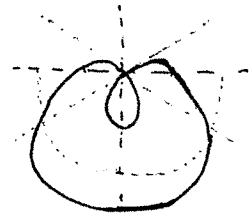
B. $r = 2\cos\theta$



C. $r = \sin 2\theta$



D. $r = 1 - 2\sin\theta$



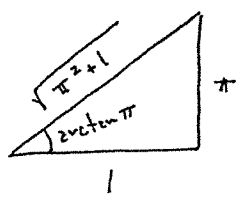
See 9.c for the upside down version

7. A. $r = 3$

$r' = 0$ so $ds = \sqrt{9+0}\, d\theta = 3\, d\theta$ $\quad S = \int_0^{2\pi} 3\, d\theta = 6\pi$

B. $r = \theta$

$\frac{dr}{d\theta} = r' = 1$ so $ds = \sqrt{\theta^2+1}\, d\theta$ so $S = \int_0^{\pi} \sqrt{\theta^2+1}\, d\theta = \int_0^{\arctan \pi} \sec^3\phi\, d\phi$



so $\sec(\arctan \pi) = \sqrt{\pi^2+1}$

Let $\theta = \tan\phi$

$d\theta = \sec^2\phi\, d\phi$

$\theta \longmapsto \phi$

$0 \quad 0$

$\pi \quad \arctan(\pi)$

$$= \frac{1}{2}\left(\sec\phi\tan\phi + \ln|\sec\phi + \tan\phi|\right)\Big|_0^{\arctan(\pi)}$$

$$= \frac{1}{2}\left(\pi\sqrt{\pi^2+1} + \ln\left(\pi + \sqrt{\pi^2+1}\right)\right)$$

C. $r = \theta^2$

$r' = 2\theta$ so $ds = \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \theta\sqrt{\theta^2+4}\, d\theta$

so $S = \int_0^{\pi} \theta\sqrt{\theta^2+4}\, d\theta = \frac{1}{2}\int_4^{\pi^2+4} u^{1/2}\, du = \frac{1}{3}u^{3/2}\Big|_4^{\pi^2+4}$

$u = \theta^2 + 4$

$du = 2\theta\, d\theta$

$$= \frac{1}{3}\left[(\pi^2+4)^{3/2} - 8\right]$$

D. $r = \theta^2 - 1$

$r' = 2\theta$ so $ds = \sqrt{(\theta^2-1)^2 + (2\theta)^2}\, d\theta$

$= \sqrt{\theta^4 - 2\theta^2 + 1 + 4\theta^2}\, d\theta$

$= \sqrt{\theta^4 + 2\theta^2 + 1}\, d\theta$

$= \sqrt{(\theta^2+1)^2}\, d\theta$

so $S = \int_0^{\pi} (\theta^2+1)\, d\theta = \frac{\theta^3}{3} + \theta\Big|_0^{\pi}$

$$= \frac{\pi^3}{3} + \pi$$

7.E. $r = e^{2\theta}$

$r' = 2e^{2\theta}$

so $ds = \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta$

$= \sqrt{5}\, e^{2\theta}\, d\theta$

$$S = \int_0^{\pi} \sqrt{5}\, e^{2\theta}\, d\theta = \frac{\sqrt{5}}{2} e^{2\theta} \Big|_0^{\pi} = \frac{\sqrt{5}}{2}\left(e^{2\pi} - 1\right)$$

F. $r = \cos^2\theta$

$r' = -2\cos\theta \sin\theta$

$(r)^2 + (r')^2 = \cos^4\theta + 4\cos^2\theta \sin^2\theta$

$= \cos^2\theta\left(\cos^2\theta + 4\sin^2\theta\right)$ ← $4\sin^2\theta = \sin^2\theta + 3\sin^2\theta$

$= \cos^2\theta\left(1 + 3\sin^2\theta\right)$

so $$S = 4\int_0^{\pi/2} \cos\theta \sqrt{1 + 3\sin^2\theta}\, d\theta = \frac{2}{\sqrt{3}}\left(2\sqrt{3} + \ln\left|2 + \sqrt{3}\right|\right)$$ ← see #1.R

Symmetry

8.A $f(x) = \begin{cases} (1+x) & -\pi \le x < 0 \\ 1 - x & 0 \le x < \pi \end{cases}$

f is <u>even</u>

$$a_0 = \frac{1}{\pi}\int_0^{\pi} (1-x)\, dx = \frac{1}{\pi}\left(x - \frac{x^2}{2}\right)\Big|_0^{\pi} = 1 - \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (1-x)\cos(nx)\, dx = \frac{2}{\pi}\left[\cancelto{0}{(1-x)\left(\frac{1}{n}\sin nx\right)\Big|_0^{\pi}} + \int_0^{\pi} \frac{1}{n}\sin nx\, dx\right]$$

$u = 1 - x$ $\quad dv = \cos nx\, dx$

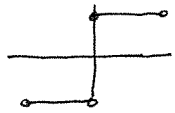
$du = -dx$ $\quad v = \frac{1}{n}\sin nx$

$$= \frac{2}{\pi}\left(-\frac{1}{n^2}\cos(nx)\right)\Big|_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left(1 - \cos n\pi\right)$$

$b_n = 0$ since f is even

8 (B) $f(x) = \begin{cases} -1 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$



f is odd

$$\left.\begin{aligned} a_0 &= 0 \\ a_n &= 0 \end{aligned}\right\} \text{ since } f \text{ is odd}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}\, dx$$

since $f(x)$ & $\sin nx$ are both odd,

$f(x)\sin(nx)$ is even

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \qquad \text{by symmetry}$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = -\frac{2}{n\pi}\cos(nx)\Big|_0^{\pi} = \frac{2}{n\pi}\left(1-\cos(n\pi)\right)$$

$$f(x) = \begin{cases} 2e^{x} & -\pi \le x \le 0 \\ 2e^{-x} & 0 < x \le \pi \end{cases}$$

f is even, so $b_n = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} e^{-x}\,dx = -\frac{2}{\pi}e^{-x}\Big|_{0}^{\pi} = \frac{2}{\pi}\left(1-e^{-\pi}\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{4}{\pi}\int_{0}^{\pi} e^{-x}\cos nx = \frac{4}{\pi}\int_{-\pi}^{0} e^{x}\cos nx\,dx$$

$$\int e^{x}\cos nx\,dx = e^{x}\cos nx + \int n\sin nx\, e^{x}\,dx = e^{x}\cos nx + n e^{x}\sin nx - \int n^2\cos nx\, e^{x}\,dx$$

$u = \cos nx \qquad dv = e^{x}dx$

$du = -n\sin nx\,dx \qquad v = e^{x}$

$u = n\sin nx \qquad dv = e^{x}\,dx$

$du = n^2\cos nx\,dx \qquad v = e^{x}$

$$(1+n^2)\int e^{x}\cos nx\,dx = e^{x}\left(\cos nx + n\sin nx\right) + C$$

$$\text{so } \frac{4}{\pi}\int_{-\pi}^{0} e^{x}\cos nx\,dx = \frac{4}{\pi(1+n^2)}e^{x}\left(\cos nx + n\sin nx\right)\Big|_{-\pi}^{0}$$

$$= \frac{4}{\pi(1+n^2)}\left[1 - e^{-\pi}\left(\cos n\pi\right)\right]$$

$$f(x) = \begin{cases} 1 & -\pi \le x < 0 \\ x & 0 \le x \le \pi \end{cases}$$

$$a_0 = \left(\int_{-\pi}^{0} 1 \, dx + \int_{0}^{\pi} x \, dx \right) \frac{1}{2\pi} = \left(\pi + \frac{\pi^2}{2} \right) \frac{1}{2\pi} = \frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4}$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx \right]$$

$u = x \qquad dv = \cos nx \, dx$

$du = dx \qquad v = \frac{1}{n} \sin nx$

$$= \frac{1}{\pi} \left[\cancelto{0}{\frac{1}{n} \sin nx \Big|_{-\pi}^{0}} + \cancelto{0}{\frac{x}{n} \sin nx \Big|_{0}^{\pi}} - \frac{1}{n} \int_{0}^{\pi} \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left(-\frac{1}{n} \left(-\frac{1}{n} \cos nx \right) \Big|_{0}^{\pi} \right) = + \frac{1}{n^2 \pi} \left(\cos n\pi - 1 \right) = \frac{-1 + \cos n\pi}{n^2 \pi} = \frac{\cos(n\pi) - 1}{n^2 \pi}$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx \right]$$

$u = x \qquad dv = \sin nx \, dx$

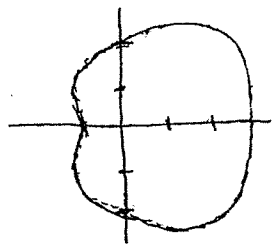
$du = dx \qquad v = -\frac{1}{n} \cos nx$

$$= \frac{1}{\pi} \left[-\frac{1}{n} \cos nx \Big|_{-\pi}^{0} + x \left(-\frac{1}{n} \cos nx \right) \Big|_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{-1}{n} + \frac{1}{n} \cos(n\pi) + \pi \left(-\frac{1}{n} \cos n\pi \right) + \cancelto{0}{\frac{1}{n^2} \sin nx \Big|_{0}^{\pi}} \right]$$

$$= \frac{1}{n\pi} \left[-1 + \cos(n\pi) - \pi \cos(n\pi) \right]$$

9. A. Inside $r = 2 + \cos\theta$

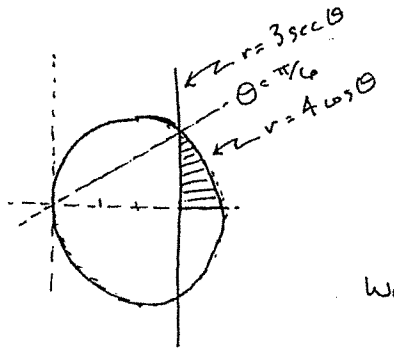


$$\frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(4 + 4\cos\theta + \cos^2\theta\right) d\theta$$

$= \frac{1}{2}(1 + \cos 2\theta)$

$$= \frac{1}{2}\left(4\theta + 4\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{2\pi} = \frac{9\pi}{2}$$

B.



Intersect when $3\sec\theta = 4\cos\theta$

$\frac{3}{4} = \cos^2\theta$

$\frac{\sqrt{3}}{2} = \cos\theta$

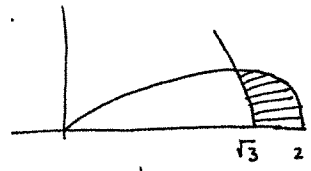
so $\theta = \pi/6$

We consider only the top half, by symmetry

$$2 \cdot \frac{1}{2}\int_0^{\pi/6} \left(4^2\cos^2\theta - 9\sec^2\theta\right) d\theta = \int_0^{\pi/6} \left(8 + 8\cos 2\theta - 9\sec^2\theta\right) d\theta$$

$$= 8\theta + 4\sin 2\theta - 9\tan\theta = \frac{4\pi}{3} + 4\,\frac{\sqrt{3}}{2} - \frac{9}{\sqrt{3}} = \frac{4\pi}{3} - \sqrt{3}$$

10. A.



$r=\sqrt{3}$ & $r=2\cos 3\theta$ intersect when $\frac{\sqrt{3}}{2}=\cos 3\theta$

so $3\theta = \pi/6$

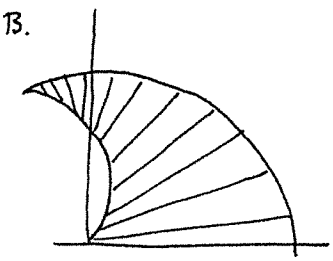
so $\theta = \pi/18$

symmetry

$$6\cdot\frac{1}{2}\int_0^{\pi/18}\left(4\cos^2 3\theta - 3\right)d\theta = 3\int_0^{\pi/18}\left(2+2\cos 6\theta - 3\right)d\theta = 3\left(-\theta+\frac{1}{3}\sin 6\theta\right)\Big|_0^{\pi/18}$$

$$= 3\left(-\frac{\pi}{18}+\frac{1}{3}\cdot\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}-\frac{\pi}{6}$$

B.



$r=3$ & $r=2-2\cos\theta$ intersect when $-\frac{1}{2}=\cos\theta$

so $\theta = \frac{2\pi}{3}$

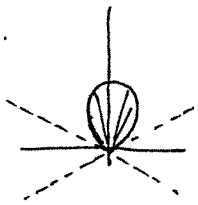
Symmetry

$$2\cdot\frac{1}{2}\int_0^{2\pi/3}\left(9-4+8\cos\theta-4\cos^2\theta\right)d\theta$$

$$=\int_0^{2\pi/3}\left(5+8\cos\theta-2-2\cos 2\theta\right)d\theta$$

$$= 3\theta+8\sin\theta-\sin 2\theta\Big|_0^{2\pi/3} = 2\pi+8\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2} = 2\pi+\frac{9\sqrt{3}}{2}$$

C.



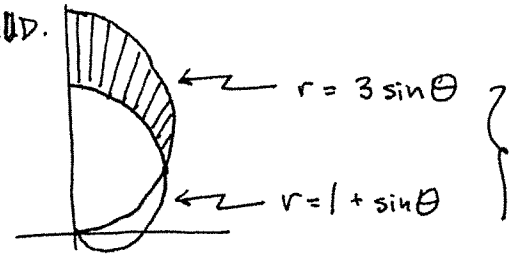
The inner loop happens when $r\le 0$, so $1+2\sin\theta\le 0$ when $\frac{7\pi}{6}\le\theta\le\frac{11\pi}{6}$

$$\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}(1+2\sin\theta)^2\,d\theta = \frac{1}{2}\int_{\frac{7\pi}{6}}^{11\pi/6}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta = \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\left(3+4\sin\theta-2\cos 2\theta\right)d\theta$$

$$=\frac{1}{2}\left(3\theta-4\cos\theta-\sin 2\theta\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} = \frac{1}{2}\left(2\pi-4\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)\right)$$

$$=\pi-\frac{3\sqrt{3}}{2}$$

10D.



intersect when

$3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = 1/2$

so $\theta = \pi/6$

Symmetry

$$2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta) \right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (4 - 4\cos 2\theta - 1 - 2\sin\theta)\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta = 3\theta - 2\sin 2\theta + 2\cos\theta \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{3\pi}{2} - \left(\frac{\pi}{2} - 2 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{\sqrt{3}}{2} \right)$$

$$= \pi$$