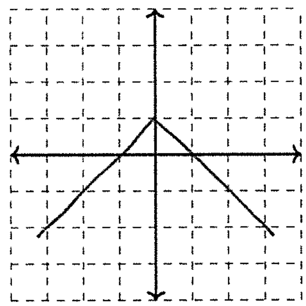
Math 172 Thanksgiving Worksheet **Name:** ____________

Sections: Fourier, 11.1 Point values in $\boxed{\text{boxes}}$.

Due: 27 November 2018

1. Let $f(x)$ be 2π-periodic with on period given by $f(x) = \begin{cases} 1+x, & -\pi \le x < 0 \\ 1-x, & 0 \le x \le \pi \end{cases}$.

   (a) $\boxed{1}$ Carefully sketch one period of $f(x)$.



   (b) $\boxed{1}$ Is $f(x)$ **odd**, **even**, or **neither**? (Circle one.)

   (c) $\boxed{6}$ Find the Fourier coefficients, a_0, a_n, and b_n for $f(x)$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} (1-x)\,dx = \frac{1}{\pi}\left(x - \frac{x^2}{2}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\pi - \frac{\pi^2}{2}\right) = 1 - \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (1-x)\cos(nx)\,dx = \frac{2}{\pi}\left[(1-x)\left[\frac{1}{n}\sin(nx)\right]\Big|_0^{\pi} + \int_0^{\pi} \frac{1}{n}\sin(nx)\,dx \right]$$

$u = (1-x)$ $\quad dv = \cos(nx)\,dx$

$du = -dx$ $\quad v = \frac{1}{n}\sin(nx)$

$$= \frac{2}{\pi}\left(-\frac{1}{n^2}\cos(nx)\right)\Big|_0^{\pi} = -\frac{2}{n^2\pi}\left(\cos(n\pi) - 1\right) = \frac{2}{n^2\pi}\left(1 - \cos(n\pi)\right)$$

$b_n = 0$ since f is even.

2. $\boxed{2}$ Eliminate the parameter t to express the parametric curve $c(t) = \left(e^{2t}, \frac{1}{1+e^{4t}}\right)$ with $t \in (-\infty, \infty)$ in the form $y = f(x)$. Include the domain of $f(x)$.

$$y = \frac{1}{1+x^2} \quad \text{for } x \in (0, \infty)$$

Parameterizing Line Segments. Since each dimension (x and y) are parameterized separately, we consider only one dimension initially.

- Consider the one dimensional line segment from $x = 0$ to $x = 1$. The standard parameterization is given by
$$x = t, t \in [0, 1].$$
- Scaling the parameterization will cause the path to be covered faster. For example, the segment from $x = 0$ to $x = 5$ can be parameterized by
$$x = 5t, t \in [0, 1].$$
- Shifting parametric equations is accomplished by adding a constant. For example, the segment from $x = -3$ to $x = 2$ can be parameterized by
$$x = 5t - 3, t \in [0, 1].$$
- Note, in the above example the form of the parameterization is $x = a + (b - a)t$ for $t \in [0, 1]$ where a is where you start and $(b - a)$ is how far you need to go.
- Note, a parameterization is not unique. The line segment from $x = -3$ to $x = 2$ can be parametrized by
$$x = 5t - 3, t \in [0, 1],$$
$$x = s, s \in [-3, 2],$$
or even
$$x = 5e^u - 3, u \in (-\infty, 0].$$
In general, I try to parameterize for $t \in [0, 1]$.
- For line segments in multiple dimensions deal with each dimension separately. For example, the line segment from $(7, 9)$ to $(0, 13)$ can be parameterized by
$$x = 7 - 7t, y = 9 + 4t, t \in [0, 1].$$
- Often our parameterization is express as a curve. For example, the parameterization above is often written $c(t) = (7 - 7t, 9 + 4t)$ for $t \in [0, 1]$.

3. $\boxed{2}$ Using the strategy above, find a parameterization of the line segment from $(1, 2)$ to $(5, -3)$. Include an appropriate domain for the parameter t.

$x = 1 + 4t$

for $0 \leq t \leq 1$

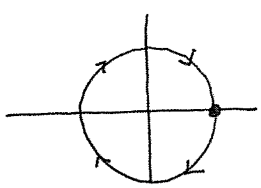
$y = 2 - 5t$

Parameterizing Circles. Since $x^2 + y^2 = R^2$ is satisfied by $c(t) = (\pm R\cos\theta, \pm R\sin\theta)$ or $c(t) = (\pm R\sin\theta, \pm R\cos\theta)$, the standard parameterization of a circle involves sines and cosines.

- A circle of radius 2 with center at the origin rotating counterclockwise and starting at $c(0) = (2, 0)$ has a standard parameterization
$$c(t) = (2\cos(t), 2\sin(t)), t \in [0, 2\pi].$$
- As with lines, we can speed up the travel along the path by scaling the parameter. A circle of radius 2 with center at the origin rotating counterclockwise and starting at $c(0) = (2, 0)$ can also be parameterized by
$$c(t) = (2\cos(2\pi t), 2\sin(2\pi t)), t \in [0, 1].$$
- By flipping the order of the sine and cosine and changing the sign on one or both, we can change the initial point ($c(0)$) and the direction of travel. For example, a circle of radius R with center at the origin rotating clockwise with initial point $c(0) = (0, -R)$ can be parameterized by
$$c(t) = (-R\sin(t), -R\cos(t)), t \in [0, 2\pi].$$
- Also as before, we can shift the center by simply adding to each component. If we take the circle above and move it so that it is centered at (x_c, y_c), we have a parameterization of
$$c(t) = (x_c - R\sin(t), y_c - R\cos(t)), t \in [0, 2\pi].$$
- An important note is that the center (x_c, y_c) is allowed to move. An important example is the cycloid (motion of a point of the circumference of a rolling circle of radius R). We track the center as $(x_c, y_c) = (R\theta, R)$. Substituting into the above we have
$$c(t) = (R\theta - R\sin(\theta), R - R\cos(\theta)).$$

  See the Desmos project on my webpage.

4. $\boxed{2}$ Find a parameterization of a circle of radius 3, center $(1, -2)$, initial point $c(0) = (4, -2)$, and drawn out in a clockwise direction as t increases.



$x = 3\cos t + 1$

$y = -3\sin t - 2$

5. $\boxed{2}$ Assume the Earth rotates in a counterclockwise direction in a circular orbit of radius 4 about the sun (located at the origin). Assume the moon rotates in a counterclockwise direction in a circular orbit about the Earth with radius 1 and completes 12 revolutions in the time the Earth completes one. Find a parameterization of the path of the moon. See the Desmos project linked on my webpage.

$x = 4\cos t + \cos(12t)$

$y = 4\sin t + \sin(12t)$

Derivatives of Parametric Curves. There are now three derivatives of interest to us. $\frac{dx}{dt}$ tells us about horizontal motion, i.e., how does x change with respect to the parameter t. $\frac{dy}{dt}$ tells us about vertical motion, i.e., how does y change with respect to the parameter t. The Chain Rule allows us to find the standard slope by computing $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$.

6. Consider the prolate cycloid given by

$$\begin{aligned} x(t) &= t - 2\sin t \\ y(t) &= 1 - 2\cos t. \end{aligned} \tag{1}$$

See the Desmos project linked on my webpage.

(a) $\boxed{1}$ For what values of $t \in [0, 2\pi)$ is $\frac{dx}{dt} = 0$? Note, these correspond to vertical tangent lines (provided $\frac{dy}{dt} \neq 0$).

$\frac{dx}{dt} = 1 - 2\cos t = 0$ when $\cos t = \frac{1}{2}$ $t = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

(b) $\boxed{1}$ For what values of $t \in [0, 2\pi)$ is $\frac{dx}{dt} < 0$? What does this tell us about the direction of travel?

$t \in [0, \frac{\pi}{3})$ or $t \in (\frac{5\pi}{3}, 2\pi)$

During these times the direction of travel is to the left, i.e., the negative x-direction.

(c) $\boxed{1}$ For what values of $t \in [0, 2\pi)$ is $\frac{dy}{dt} = 0$? How have we referred to these points in the past?

$\frac{dy}{dt} = 2\sin t = 0$ when $t = 0$ or π or ...

These are critical points.

(d) $\boxed{1}$ Find the slope of the curve, $\frac{dy}{dx}$, at $t = \pi/6$.

$\frac{dy}{dx} = \frac{2\sin t}{1 - 2\cos t}$ so $\left.\frac{dy}{dx}\right|_{t=\pi/6} = \frac{2(\frac{1}{2})}{1 - 2(\frac{\sqrt{3}}{2})} = \frac{1}{1-\sqrt{3}}$