1. Let \( f(x) \) be \( 2\pi \)-periodic with on period given by
\[
\begin{cases} 
1 + x, & -\pi \leq x < 0 \\
1 - x, & 0 \leq x \leq \pi 
\end{cases}
\]

(a) Carefully sketch one period of \( f(x) \).

(b) Is \( f(x) \) odd, even, or neither? (Circle one.)

(c) Find the Fourier coefficients, \( a_0, a_n, \) and \( b_n \) for \( f(x) \).

2. Eliminate the parameter \( t \) to express the parametric curve 
\[
c(t) = \left( e^{2t}, \frac{1}{1 + e^{4t}} \right)
\]

with \( t \in (-\infty, \infty) \) in the form \( y = f(x) \). Include the domain of \( f(x) \).
Parameterizing Line Segments. Since each dimension \((x \text{ and } y)\) are parameterized separately, we consider only one dimension initially.

- Consider the one dimensional line segment from \(x = 0\) to \(x = 1\). The standard parameterization is given by
  \[
x = t, t \in [0, 1].
  \]
- Scaling the parameterization will cause the path to be covered faster. For example, the segment from \(x = 0\) to \(x = 5\) can be parameterized by
  \[
x = 5t, t \in [0, 1].
  \]
- Shifting parametric equations is accomplished by adding a constant. For example, the segment from \(x = -3\) to \(x = 2\) can be parameterized by
  \[
x = 5t - 3, t \in [0, 1].
  \]
- Note, in the above example the form of the parameterization is \(x = a + (b - a)t\) for \(t \in [0, 1]\) where \(a\) is where you start and \((b - a)\) is how far you need to go.
- Note, a parameterization is not unique. The line segment from \(x = -3\) to \(x = 2\) can be parametrized by
  \[
x = 5t - 3, t \in [0, 1],
  \]
  \[
x = s, s \in [-3, 2],
  \]
  or even
  \[
x = 5e^u - 3, u \in (-\infty, 0].
  \]
  In general, I try to parameterize for \(t \in [0, 1]\).
- For line segments in multiple dimensions deal with each dimension separately. For example, the line segment from \((7, 9)\) to \((0, 13)\) can be parameterized by
  \[
x = 7 - 7t, y = 9 + 4t, t \in [0, 1].
  \]
- Often our parameterization is express as a curve. For example, the parameterization above is often written \(c(t) = (7 - 7t, 9 + 4t)\) for \(t \in [0, 1]\).

3. Using the strategy above, find a parameterization of the line segment from \((1, 2)\) to \((5, -3)\). Include an appropriate domain for the parameter \(t\).
Parameterizing Circles. Since \( x^2 + y^2 = R^2 \) is satisfied by \( c(t) = (\pm R \cos \theta, \pm R \sin \theta) \) or \( c(t) = (\pm R \sin \theta, \pm R \cos \theta) \), the standard parameterization of a circle involves sines and cosines.

- A circle of radius 2 with center at the origin rotating counterclockwise and starting at \( c(0) = (2, 0) \) has a standard parameterization
  \[
  c(t) = (2 \cos(t), 2 \sin(t)), t \in [0, 2\pi].
  \]

- As with lines, we can speed up the travel along the path by scaling the parameter. A circle of radius 2 with center at the origin rotating counterclockwise and starting at \( c(0) = (2, 0) \) can also be parameterized by
  \[
  c(t) = (2 \cos(2\pi t), 2 \sin(2\pi t)), t \in [0, 1].
  \]

- By flipping the order of the sine and cosine and changing the sign on one or both, we can change the initial point \( (c(0)) \) and the direction of travel. For example, a circle of radius R with center at the origin rotating clockwise with initial point \( c(0) = (0, -R) \) can be parameterized by
  \[
  c(t) = (-R \sin(t), -R \cos(t)), t \in [0, 2\pi].
  \]

- Also as before, we can shift the center by simply adding to each component. If we take the circle above and move it so that it is centered at \((x_c, y_c)\), we have a parameterization of
  \[
  c(t) = (x_c - R \sin(t), y_c - R \cos(t)), t \in [0, 2\pi].
  \]

- An important note is that the center \((x_c, y_c)\) is allowed to move. An important example is the cycloid (motion of a point of the circumference of a rolling circle of radius \( R \)). We track the center as \( (x_c, y_c) = (R\theta, R) \). Substituting into the above we have
  \[
  c(t) = (R\theta - R \sin(\theta), R - R \cos(\theta)).
  \]

See the Desmos project on my webpage.

4. Find a parameterization of a circle of radius 3, center \((1, -2)\), initial point \( c(0) = (4, -2) \), and drawn out in a clockwise direction as \( t \) increases.

5. Assume the Earth rotates in a counterclockwise direction in a circular orbit of radius 4 about the sun (located at the origin). Assume the moon rotates in a counterclockwise direction in a circular orbit about the Earth with radius 1 and completes 12 revolutions in the time the Earth completes one. Find a parameterization of the path of the moon. See the Desmos project linked on my webpage.
Derivatives of Parametric Curves. There are now three derivatives of interest to us. \( \frac{dx}{dt} \) tells us about horizontal motion, i.e., how does \( x \) change with respect to the parameter \( t \). \( \frac{dy}{dt} \) tells us about vertical motion, i.e., how does \( y \) change with respect to the parameter \( t \). The Chain Rule allows us to find the standard slope by computing \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \).

6. Consider the prolate cycloid given by

\[
\begin{align*}
  x(t) &= t - 2 \sin t \\
  y(t) &= 1 - 2 \cos t.
\end{align*}
\]  

(1)

See the Desmos project linked on my webpage.

(a) For what values of \( t \in [0, 2\pi) \) is \( \frac{dx}{dt} = 0 \)? Note, these correspond to vertical tangent lines (provided \( \frac{dy}{dt} \neq 0 \)).

(b) For what values of \( t \in [0, 2\pi) \) is \( \frac{dx}{dt} < 0 \)? What does this tell us about the direction of travel?

(c) For what values of \( t \in [0, 2\pi) \) is \( \frac{dy}{dt} = 0 \)? How have we referred to these points in the past?

(d) Find the slope of the curve, \( \frac{dy}{dx} \), at \( t = \pi/6 \).