Exam 2 Review - Solutions

1. A. $\displaystyle\int_0^{\pi} \sin^3 5x \, dx = \int_0^{\pi} (1-\cos^2 5x) \sin 5x \, dx = -\frac{1}{5}\int_1^{-1} (1-u^2)\,du = \frac{1}{5}\int_{-1}^{1} (1-u^2)\,du$

let $u = \cos 5x$

$du = -5 \sin 5x$

$x = 0 \longmapsto u = 1$

$x = \pi \longmapsto u = -1$

$$= -\frac{1}{5}\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} = \frac{1}{5}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right] \leftarrow \text{Stop here}$$

$$= \frac{1}{5}\cdot\frac{4}{3} = \frac{4}{15}$$

B. $\displaystyle\int \sin^9 x \cos^3 x \, dx = \int \sin^9 x (1-\sin^2 x) \cos x \, dx = \int u^9 (1-u^2)\,du$

let $u = \sin x$

$du = \cos x \, dx$

$$= \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{12} + C$$

C. $\displaystyle\int \sin^2 \pi x \, dx = \int \frac{1}{2}(1-\cos 2\pi x)\,dx = \frac{1}{2}\left(x - \frac{1}{2\pi}\sin 2\pi x\right) + C$

D. $\displaystyle\int \tan^2 x \sec^2 x \, dx = \frac{1}{3}\tan^3 x + C$

$u = \tan x \quad du = \sec^2 x \, dx$

E. $\displaystyle\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x (1+\tan^2 x)\sec^2 x \, dx = \int u^6 (1+u^2)\,du$

$u = \tan x \quad du = \sec^2 x \, dx$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

F. $\displaystyle\int \sec^3 2x \tan^3 2x \, dx = \int \sec^2 2x (\sec^2 2x - 1) \sec 2x \tan 2x \, dx$

$u = \sec 2x \quad du = 2\sec 2x \tan 2x \, dx$

$$= \frac{1}{2}\int u^2 (u^2 - 1)\,du = \frac{1}{2}\int (u^4 - u^2)\,du$$

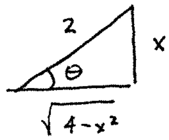
$$= \frac{\sec^5 2x}{10} - \frac{\sec^3 2x}{6} + C$$

2. A. $\int \sqrt{4-x^2}\,dx = \int 2\cos\theta \cdot 2\cos\theta\,d\theta = \int 2(1+\cos 2\theta)\,d\theta$

Given: $\sin 2\theta = 2\sin\theta\cos\theta$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$



$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = 2(\theta + \sin\theta\cos\theta) + C$

$= 2\left(\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right) + C$

B. $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \int_0^{\pi/6} \frac{\sqrt{3}\tan\theta}{\cancel{\sqrt{3}}\,\cancel{\sec\theta}}\,\cancel{\sqrt{3}}\,\cancel{\sec\theta}\tan\theta\,d\theta = \sqrt{3}\int_0^{\pi/6} \tan^2\theta\,d\theta$

Let $x = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

$x = \sqrt{3} \mapsto \theta = 0$

$x = 2 \mapsto \theta = \pi/6$

$= \sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta$

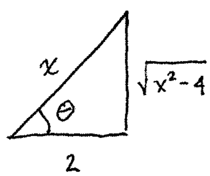
$= \sqrt{3}(\tan\theta - \theta)\Big|_0^{\pi/6} = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$

C. $\int \frac{x^2}{\sqrt{x^2-4}}\,dx = \int \frac{4\sec^2\theta}{\cancel{2}\cancel{\tan\theta}} \cdot \cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta = 4\int \sec^3\theta\,d\theta$

This is a given integral

Let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$



$= 2\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$

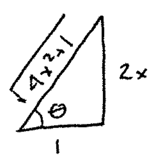
$= 2\left(\frac{x}{2}\cdot\frac{\sqrt{x^2-4}}{2} + \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right|\right) + C$

D. $\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{\sec\theta} = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + c$

given

Let $2x = \tan\theta$

$2\,dx = \sec^2\theta\,d\theta$

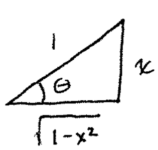


$= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C$

E. $\int \frac{x^3}{\sqrt{1-x^2}}\,dx = = \int \frac{\sin^3\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta = -\int (1-u^2)\,du$

Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$



$u = \cos\theta$

$du = -\sin\theta\,d\theta$

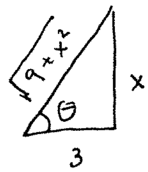
$= \frac{u^3}{3} - u + C = \frac{\cos^3\theta}{3} - \cos\theta + C$

$= \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + C$

2. F. $\displaystyle\int \frac{9}{(9+x^2)^2}\,dx = \int \frac{\cancel{9}\cdot 3\sec^2\theta}{\cancel{9}\cdot 9\sec^4\theta}\,d\theta = \frac{1}{3}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{3}\int \cos^2\theta\,d\theta$

Let $x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$



$\displaystyle = \frac{1}{6}\int (1+\cos 2\theta)\,d\theta = \frac{1}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$\sin 2\theta = 2\sin\theta\cos\theta$, given

$\displaystyle = \frac{1}{6}\left(\theta + \sin\theta\cos\theta\right) + C$

$\displaystyle = \frac{1}{6}\left(\arctan\left(\frac{x}{3}\right) + \frac{3x}{9+x^2}\right) + C$

3. A. $\displaystyle\int \frac{3-x}{1-x^2}\,dx = \int \left(\frac{1}{1-x} + \frac{2}{1+x}\right)dx$

$= -\ln|1-x| + 2\ln|1+x| + C$

$\displaystyle\frac{3-x}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

$3 - x = A(1+x) + B(1-x)$

Let $x = 1$, $2 = A(2)$ so $A = 1$

Let $x = -1$, $4 = B(2)$ so $B = 2$

B. $\displaystyle\int_{-1}^{0} \frac{3}{x^2+x-2}\,dx = \int_{-1}^{0}\left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx = \ln|x-1| - \ln|x+2|\Big|_{-1}^{0}$

$= \ln|-1| - \ln|2| - \ln|-2| + \ln|1|$

$= -2\ln(2)$

$\displaystyle\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$3 = A(x+2) + B(x-1)$

Let $x = 1$, $3 = A(3)$ so $A = 1$

Let $x = -2$, $3 = B(-3)$ so $B = -1$

3.C. $\displaystyle\int \frac{3x^2+x+9}{x^3+9x}\,dx = \int\left(\frac{1}{x}+\frac{2x+1}{x^2+9}\right)dx = \int\left(\frac{1}{x}+\frac{2x}{x^2+9}+\frac{1}{x^2+9}\right)dx$

$$= \ln|x| + \ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$\frac{3x^2+x+9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$3x^2+x+9 = A(x^2+9) + (Bx+C)x$$

Let $x=0$, $9 = A(9) \Rightarrow A=1$

Equate coeff: x^2: $3 = A+B \Rightarrow B=2$

x: $1 = C$

3.D. $\displaystyle\int \frac{4x^2+5x+3}{x(x+1)^2}\,dx = \int\left(\frac{3}{x}+\frac{1}{x+1}-\frac{2}{(x+1)^2}\right)dx$

$$= 3\ln|x| + \ln|x+1| + \frac{2}{x+1} + C$$

$$\frac{4x^2+5x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$4x^2+5x+3 = A(x+1)^2 + Bx(x+1) + Cx$$

let $x=0$, $3 = A$

Equate coeff: x^2: $4 = A+B \Rightarrow B=1$

x: $5 = 2A+B+C \Rightarrow C=-2$

3.E. $\displaystyle\int \frac{x^3-4x^2+10x}{x^2-4x+8}\,dx = \int\left(x+\frac{2x}{x^2-4x+8}\right)dx = \int\left(x+\frac{2x-4}{x^2-4x+8}+\frac{4}{x^2-4x+8}\right)dx$

split so that we can do this u-sub on the middle part

$u = x^2-4x+8$
$du = (2x-4)\,dx$

$$\begin{array}{r} x \\ x^2-4x+8 \overline{)\, x^3-4x^2+10x} \\ \underline{x^3-4x^2+8x} \\ 2x \end{array}$$

$$= \int\left(x + \frac{2x-4}{x^2-4x+8} + \frac{4}{(x-2)^2+2^2}\right)dx$$

$$= \frac{x^2}{2} + \ln|x^2-4x+8| + 2\arctan\left(\frac{x-2}{2}\right) + C$$

F. $\displaystyle\int \frac{\sin x}{\cos x - \cos^2 x}\,dx = -\int \frac{du}{u-u^2} = \int \frac{du}{u^2-u} = \int\left(\frac{1}{u-1} - \frac{1}{u}\right)du = \ln|u-1| - \ln|u| + C$

$= \ln|\cos x - 1| - \ln|\cos x| + C$

$u = \cos x$

$du = -\sin x\,dx$

$$\frac{1}{u^2-u} = \frac{A}{u-1} + \frac{B}{u}$$

$$1 = Au + B(u-1)$$

Let $u=0$, $1 = B(-1) \Rightarrow B=-1$

$u=1$, $1 = A \Rightarrow A=1$

4. A. $\displaystyle\int_{-\infty}^{0} xe^x\,dx = xe^x\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^x\,dx = (xe^x - e^x)\Big|_{-\infty}^{0} = \lim_{R\to-\infty}(xe^x - e^x)\Big|_R^0$

$u = x$ $\quad dv = e^x\,dx$

$du = dx$ $\quad v = e^x$

$$= \lim_{R\to-\infty}\left(0 - 1 - Re^R + e^R\right)$$

$$= -1 - \lim_{R\to-\infty}\left(Re^R\right)$$

$(-\infty)(0)$ form which is indeterminate, so we use L'Hopital's Rule

$$= -1 - \lim_{R\to-\infty}\left(\frac{R}{e^{-R}}\right)$$

$\frac{(-\infty)}{\infty}$ form, so L'H applies

$$\overset{L'H}{=} -1 - \lim_{R\to-\infty}\left(\frac{1}{-e^{-R}}\right) = -1$$

B. $\displaystyle\int_4^\infty \frac{5}{x^2-x-6}\,dx = \int_4^\infty\left(\frac{1}{x-3} - \frac{1}{x+2}\right)dx = \lim_{R\to\infty}\left(\ln|x-3| - \ln|x+2|\right)\Big|_4^R$

$$\frac{5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$5 = A(x+2) + B(x-3)$$

Let $x=3$, $5 = 5A \Rightarrow A=1$

$x=-2$, $5 = -5B \Rightarrow B=-1$

$$= \lim_{R\to\infty}\Big(\underbrace{\ln|R-3| - \ln|R+2|}_{\infty - \infty \text{ indeterminate form}} - \ln|4-3| + \ln|4+2|\Big)$$

$$= \lim_{R\to\infty} \ln\left|\frac{R-3}{R+2}\right| + \ln 6$$

$$= \ln\left(\lim_{R\to\infty}\frac{R-3}{R+2}\right) + \ln 6 = \ln 6$$

Since $\ln(\cdot)$ is continuous, the limit & the function can switch.

4. c. $$\int_0^\infty \frac{2}{3x+5}\,dx = \lim_{R\to\infty} \frac{2}{3}\ln|3x+5|\Big|_0^R = \lim_{R\to\infty} \frac{2}{3}\ln|3R+5| - \frac{2}{3}\ln 5 = \infty$$

so the integral diverges.

D. $$\int_0^\infty \frac{\arctan x}{x^2+1}\,dx = \lim_{R\to\infty} \frac{1}{2}\arctan^2 x\Big|_0^R = \lim_{R\to\infty} \frac{1}{2}\arctan^2 R - 0 = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$$

E. $$\int_0^1 \ln x\,dx = x\ln x\Big|_0^1 - \int_0^1 dx = \lim_{R\to0^+} (x\ln x - x)\Big|_R^1 = \lim_{R\to0^+} \left(-1 - R\ln R + R\right)$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$= -1 - \lim_{R\to0^+} (R\ln R) \longleftarrow 0\cdot(-\infty) \text{ indeterminate form}$$

$$= -1 - \lim_{R\to0^+} \left(\frac{\ln R}{1/R}\right) \longleftarrow \frac{-\infty}{\infty} \text{ indeterminate form, L'H applies}$$

$$\overset{L'H}{=} -1 - \lim_{R\to0^+} \left(\frac{1/R}{-1/R^2}\right) = -1 - \lim_{R\to0^+}(-R) = -1$$

F. $$\int_{-5}^{2} \frac{dx}{\sqrt[5]{x-2}} = \int_{-7}^{0} u^{-1/5}\,du = \lim_{R\to0^-} \frac{5}{4}u^{4/5}\Big|_{-7}^{R} = \lim_{R\to0^-} \left(\frac{5}{4}(R)^{4/5} - \frac{5}{4}(-7)^{4/5}\right)$$

Let $u = x-2$

$du = dx$

$x=-5 \mapsto u=-7$

$x\to2 \mapsto u\to0$

$$= -\frac{5}{4}(-7)^{4/5}$$

5. A. $\displaystyle\int_1^\infty \frac{dx}{x^4+x+7}$

We have $0 \leq \dfrac{1}{x^4+x+7} \leq \dfrac{1}{x^4}$ for $x \geq 1$.

Additionally, the p-integral $\displaystyle\int_1^\infty \frac{dx}{x^4}$ converges since $p = 4 > 1$.

So, by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{dx}{x^4+x+7}$ also converges.

B. $\displaystyle\int_2^\infty \frac{2}{x-\sqrt{x}}\,dx$

Since $0 \leq \dfrac{2}{x} \leq \dfrac{2}{x-\sqrt{x}}$ for $x \geq 2$ & $\displaystyle\int_2^\infty \frac{2}{x}\,dx = 2\int_2^\infty \frac{dx}{x}$ is a divergent

p-integral $(p = 1 \leq 1)$, by comparison, $\displaystyle\int_2^\infty \frac{2}{x-\sqrt{x}}\,dx$ also diverges.

C. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges by comparison since $0 \leq \dfrac{\cos x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$ for $x \in (0, \pi/2]$

and $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is a convergent p-integral $(p = 1/2 < 1)$

<u>Note</u>: in the three arguments above, I wrote them in different orders. The important thing is that each contained three parts.

1) A comparison between <u>functions</u> (not integrals), e.g. $0 \leq \dfrac{\cos x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$ for $x \in (0, \frac{\pi}{2}]$

2) Justification that an <u>integral</u> (not function) converges/diverges, ← This will almost always reference a p-integral.
e.g. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is a convergent p-integral $(p = 1/2 < 1)$

3) A conclusion about an <u>integral</u> (not a function)
e.g. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges by comparison.

5.D. $\displaystyle\int_0^2 \frac{e^{x^2}}{x^2}\,dx$

Initially note that for $x \in (0,2]$ $\quad 0 \leq \frac{1}{x^2} \leq \frac{e^{x^2}}{x^2}$.

Additionally, the p-integral $\displaystyle\int_0^2 \frac{dx}{x^2}$ diverges $(p = 2 \geqslant 1)$.

So, by the Comparison Theorem $\displaystyle\int_0^2 \frac{e^{x^2}}{x^2}\,dx$ also diverges.

E. $\displaystyle\int_2^\infty \frac{e^{-x^2}}{x^2+1}\,dx$

Since $e^{-x^2} \leq 1$ for all x, we have $0 \leq \frac{e^{-x^2}}{x^2+1} \leq \frac{1}{x^2}$.

Now, the p-integral $\displaystyle\int_2^\infty \frac{dx}{x^2}$ converges since $p = 2 > 1$,

so by the Comparison Theorem $\displaystyle\int_2^\infty \frac{e^{-x^2}}{x^2+1}\,dx$ also converges.

F. $\displaystyle\int_0^2 \frac{dx}{x^2+\sqrt{x}}$

For $x \in (0,2]$ we have $0 \leq \frac{1}{x^2+\sqrt{x}} \leq \frac{1}{\sqrt{x}}$. Since $\displaystyle\int_0^2 \frac{dx}{\sqrt{x}}$ is a convergent p-integral, $(p = \frac{1}{2} < 1)$, by the Comparison Theorem $\displaystyle\int_0^2 \frac{dx}{x^2+\sqrt{x}}$ also converges.

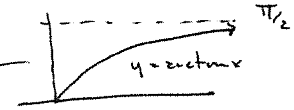
6. A. $p(x) = Cx^2$ on $[0,3]$

$$1 = \int_0^3 Cx^2\,dx = \frac{C}{3}x^3 \Big|_0^3 = 9C \Rightarrow C = \frac{1}{9}$$

$$P(1 \le X \le 2) = \int_1^2 \frac{1}{9}x^2\,dx = \frac{1}{27}x^3 \Big|_1^2 = \frac{7}{27}$$

B. $p(x) = \dfrac{C}{x^2+1}$ on $[0,\infty)$

$$1 = \int_0^\infty \frac{C}{x^2+1}\,dx = C \lim_{R\to\infty} (\arctan R - \arctan 0) = C\left(\frac{\pi}{2}\right) \Rightarrow C = \frac{2}{\pi}$$



$$P(x \ge 1) = \int_1^\infty \frac{2/\pi}{x^2+1}\,dx = \frac{2}{\pi} \lim_{R\to\infty} (\arctan R - \arctan 1) = \frac{2}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{2}$$

know this!

C. $p(x) = \dfrac{C}{\sqrt{4-x^2}}$ on $[0,2)$

$$1 = \int_0^2 \frac{C}{\sqrt{4-x^2}}\,dx = C \cdot \lim_{R\to 2^-} \arcsin\left(\frac{x}{2}\right)\Big|_0^R = C \lim_{R\to 2^-} \arcsin\left(\frac{R}{2}\right) - \arcsin(0)$$

$$= C\arcsin(1) = C\left(\frac{\pi}{2}\right) \Rightarrow C = \frac{2}{\pi}$$

$$P(0 \le X \le 1) = \int_0^1 \frac{2/\pi}{\sqrt{4-x^2}} = \frac{2}{\pi}\arcsin\left(\frac{x}{2}\right)\Big|_0^1 = \frac{2}{\pi}\left(\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi}{6}\right) = \frac{1}{3}$$

7\. A. $y = 1 + 3x \quad [0, 2]$

$y' = 3$

$1 + (y')^2 = 10$

$$S = \int_0^2 \sqrt{10}\, dx = 2\sqrt{10}$$

B. $y = \frac{1}{3}(2 + x^2)^{3/2} \quad [0, 2]$

$y' = x(2 + x^2)^{1/2}$

$1 + (y')^2 = 1 + 2x^2 + x^4$

$= (1 + x^2)^2$

$$S = \int_0^2 \sqrt{(1+x^2)^2}\, dx = \int_0^2 (1 + x^2)\, dx = x + \frac{x^3}{3} \Big|_0^2 = 2 + \frac{8}{3} = \frac{14}{3}$$

C. $y = 1 + x^2 \quad [0, 2]$

$y' = 2x$

$1 + (y')^2 = 1 + 4x^2$

$$S = \int_0^2 \sqrt{1 + 4x^2}\, dx = \int_0^{\varphi} \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\varphi}$$

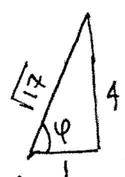
given

let $2x = \tan\theta$

$2\,dx = \sec^2\theta\, d\theta$

$x = 0 \longmapsto \theta = 0$

$x = 2 \longmapsto \theta = \underbrace{\arctan 4}_{= \varphi}$



$$= \frac{1}{2}\left(4\sqrt{17} + \ln|\sqrt{17} + 4|\right)$$

so

$\sec\varphi = \sqrt{17}$

$\tan\varphi = 4$

Use a △ to unwrap $\sec(\arctan 4)$

give the angle a name so we don't have to keep writing arctan 4

D. $y = \frac{x^4}{32} + \frac{1}{x^2} \quad [1, 2]$

$y' = \frac{x^3}{8} - \frac{2}{x^3}$

$1 + (y')^2 = 1 + \left(\frac{x^6}{64} - \frac{1}{2} + \frac{4}{x^6}\right)$

$= \frac{x^6}{64} + \frac{1}{2} + \frac{4}{x^6}$

$= \left(\frac{x^3}{8} + \frac{2}{x^3}\right)^2$

$$S = \int_1^2 \sqrt{\left(\frac{x^3}{8} + \frac{2}{x^3}\right)^2}\, dx = \int_1^2 \left(\frac{1}{8}x^3 + 2x^{-3}\right) dx$$

$$= \frac{x^4}{32} - \frac{1}{x^2}\Big|_1^2$$

$$= \left(\frac{16}{32} - \frac{1}{4}\right) - \left(\frac{1}{32} - 1\right)$$

7. E. $y = \ln \cos x \qquad [0, \pi/3]$

$y' = \dfrac{-\sin x}{\cos x} = -\tan x$

$1 + (y')^2 = 1 + \tan^2 x$

$= \sec^2 x$

$$S = \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/3} \sec x \, dx$$

given

$$= \ln |\sec x + \tan x| \Big|_0^{\pi/3}$$

$$= \ln |2 + \sqrt{3}| - \ln |1 + 0|$$

$$= \ln |2 + \sqrt{3}|$$

F. $y = \dfrac{x^5}{20} + \dfrac{1}{3x^3} \qquad [1,2]$

$y' = \dfrac{x^4}{4} - \dfrac{1}{x^4}$

$1 + (y')^2 = 1 + \left(\dfrac{x^8}{16} - \dfrac{1}{2} + \dfrac{1}{x^8} \right)$

$= \dfrac{x^8}{16} + \dfrac{1}{2} + \dfrac{1}{x^8} = \left(\dfrac{x^4}{4} + \dfrac{1}{x^4} \right)^2$

$$S = \int_1^2 \sqrt{\left(\tfrac{x^4}{4} + x^{-4}\right)^2}\, dx = \int_1^2 \left(\tfrac{1}{4} x^4 + x^{-4} \right) dx$$

$$= \frac{x^5}{20} - \frac{1}{3x^3} \Big|_1^2$$

$$= \left(\frac{32}{20} - \frac{1}{24} \right) - \left(\frac{1}{20} - \frac{1}{3} \right)$$

8. A. $y = x^3 \qquad [0,1]$

$y' = 3x^2$

$1 + (y')^2 = 1 + 9x^4$

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx = 2\pi \cdot \frac{1}{36} \int_1^{10} u^{1/2}\, du$$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

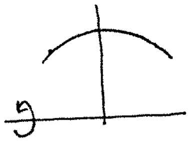
$x = 0 \longmapsto u = 1$

$x = 1 \longmapsto u = 10$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \frac{\pi}{27} \left(10^{3/2} - 1 \right)$$

8.B. $y = \sqrt{4-x^2}$ $[-1, 1]$



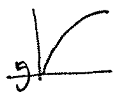
$$y' = \frac{-x}{\sqrt{4-x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{4-x^2}$$

$$= \frac{4 - x^2 + x^2}{4-x^2} = \frac{4}{4-x^2}$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\,dx = 2\pi \int_{-1}^{1} 2\,dx = 4\pi x \Big|_{-1}^{1} = 8\pi$$

C. $y = \sqrt{x}$ $[0, 1]$



$$y' = \frac{1}{2\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{4x}$$

$$= \frac{4x+1}{4x}$$

$$S = 2\pi \int_0^1 \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx = \pi \int_0^1 \sqrt{4x+1}\,dx = \frac{\pi}{4}\int_1^5 u^{1/2}\,du$$

$u = 4x+1$

$du = 4\,dx$

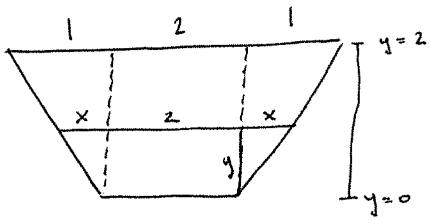
$x = 0 \longmapsto u = 1$

$x = 1 \longmapsto u = 5$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5$$

$$= \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

9.



By similar triangles $\frac{x}{y} = \frac{1}{2}$ so $x = \frac{1}{2}y$.

The width of a slice is $w = 2 + 2x = 2 + y$.

The area of a slice is $(2+y)\Delta y$.

The depth of a slice is $(2-y)$.

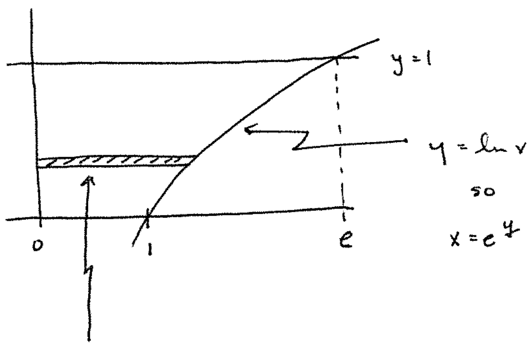
$$\text{Force} = \rho g \int_0^2 (2+y)(2-y)\,dy$$

$$= \rho g \int_0^2 (4 - y^2)\,dy$$

$$= \rho g \left(4y - \frac{y^3}{3}\right)\Big|_0^2$$

$$= \frac{16}{3}\rho g$$

10.



The area of this slice is $x \, \Delta y = e^y \, \Delta y$.

The depth of this slice is $(1-y)$.

$$\text{Force} = \rho g \int_0^1 (1-y) e^y \, dy$$

$u = (1-y) \qquad dv = e^y \, dy$

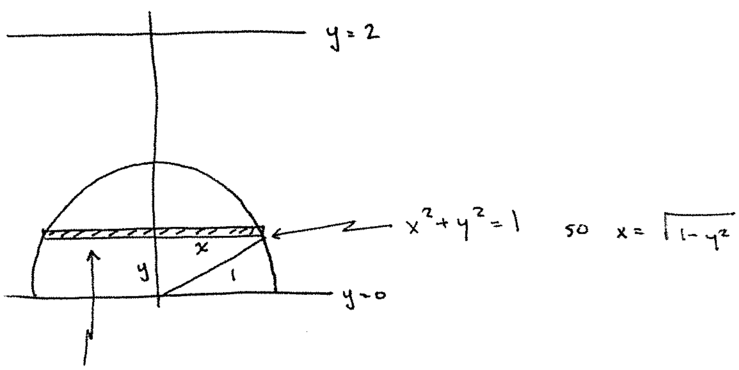
$du = -dy \qquad v = e^y$

$$= \rho g \left[(1-y)e^y \Big|_0^1 + \int_0^1 e^y \, dy \right]$$

$$= \rho g \left(0 - 1 + e^y \Big|_0^1 \right)$$

$$= \rho g \left(0 - 1 + e - 1 \right) = \rho g (e - 2)$$

11.



The area of this slice is $2x \, \Delta y = 2\sqrt{1-y^2} \, \Delta y$.

The depth of this slice is $(2-y)$.

This is the area of a quarter circle of radius 1, no need to do a trig sub.

$$\text{Force} = \rho g \int_0^1 2(2-y)\sqrt{1-y^2} \, dy = 4\rho g \int_0^1 \sqrt{1-y^2} \, dy - 2\rho g \int_0^1 y\sqrt{1-y^2} \, dy$$

$u = 1 - y^2$

$du = -2y \, dy$

$y = 0 \longmapsto u = 1$

$y = 1 \longmapsto u = 0$

$$= 4\rho g \left(\frac{\pi}{4} \right) + \rho g \int_1^0 u^{1/2} \, du$$

$$= 4\rho g \left(\frac{\pi}{4} \right) + \rho g \cdot \frac{2}{3} u^{3/2} \Big|_1^0$$

$$= \rho g \pi - \frac{2}{3}\rho g = \rho g \left(\pi - \frac{2}{3} \right)$$

12\. A. $\displaystyle\int \cos^5 3x \sin^2 3x\,dx = \int (1-\sin^2 3x)^2 \sin^2 3x \cos 3x\,dx = \frac{1}{3}\int (1-2u^2+u^4)\,u^2\,du$

let $u = \sin 3x$

$du = 3\cos 3x\,dx$

$$= \frac{1}{3}\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C = \frac{\sin^3 3x}{9} - \frac{2\sin^5 3x}{15} + \frac{\sin^7 3x}{21} + C$$

B. $\displaystyle\int \frac{x^3+x^2-9x-19}{x^2-x-6}\,dx = \int \left(x+2 - \frac{x+7}{x^2-x-6}\right)dx = \int \left(x+2 - \frac{2}{x-3} + \frac{1}{x+2}\right)dx$

$$\begin{array}{r} x+2 \\ x^2-x-6 \overline{)\,x^3+x^2-9x-19} \\ \underline{-x^3+x^2+6x} \\ 2x^2-3x-19 \\ \underline{-2x^2+2x+12} \\ -x-7 \end{array}$$

$\dfrac{x+7}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$= x^2 + 2x - 2\ln|x-3| + \ln|x+2| + C$

$x+7 = A(x+2) + B(x-3)$

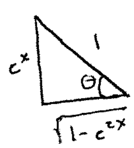
Let $x = 3$, $10 = A(5) \Rightarrow A=2$

Let $x=-2$, $5 = B(-5) \Rightarrow B=-1$

C. $\displaystyle\int e^x\sqrt{1-e^{2x}}\,dx = \int \cos^2\theta\,d\theta = \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C$

Let $e^x = \sin\theta$

$e^x\,dx = \cos\theta\,d\theta$



$$= \frac{1}{2}\left(\arcsin(e^x) + e^x\sqrt{1-e^{2x}}\right) + C$$

D. $\displaystyle\int \sec^9 x \tan^3 x\,dx = \int \sec^8 x(\sec^2 x - 1)\sec x\tan x\,dx = \int (u^{10}-u^8)\,du$

$u = \sec x$ $\quad du = \sec x\tan x\,dx$

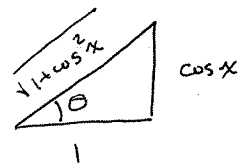
$$= \frac{1}{10}\sec^{10}x - \frac{1}{8}\sec^8 x + C$$

E. $\displaystyle\int \sin x\sqrt{1+\cos^2 x}\,dx = -\int \sec^3\theta\,d\theta = -\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C$

Let $\cos x = \tan\theta$

$-\sin x\,dx = \sec^2\theta\,d\theta$

$$= C - \frac{1}{2}\left(\cos x\sqrt{1+\cos^2 x} + \ln\left|\sqrt{1+\cos^2 x} + \cos x\right|\right) + C$$



12. F. $\displaystyle\int \frac{6x^2 - 8x + 56}{x^4 - 16}\,dx$

$$\frac{6x^2 - 8x + 56}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

$$6x^2 - 8x + 56 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x^2-4)$$

Let $x = 2$, $\quad 24 - 16 + 56 = A(4)(8) \Rightarrow 64 = 32A \Rightarrow A = 2$

Let $x = -2$, $\quad 24 + 16 + 56 = B(-4)(8) \Rightarrow 96 = -32B \Rightarrow B = -3$

Equate coeff: $\quad x^3: \; 0 = A + B + C \Rightarrow C = 1$

$\quad x^0: \; 56 = 8A - 8B - 4D \Rightarrow D = -4$

So $$\int \left(\frac{2}{x-2} - \frac{3}{x+2} + \frac{x}{x^2+4} - \frac{4}{x^2+4} \right) dx$$

$$= 2\ln|x-2| - 3\ln|x+2| + \frac{1}{2}\ln|x^2+4| - 2\arctan\left(\frac{x}{2}\right) + C$$

12. G. $$\int \sec^6 4x\,dx = \int (1+\tan^2 4x)^2 \sec^2 4x\,dx = \frac{1}{4}\int (1 + 2u^2 + u^4)\,du$$

$u = \tan 4x$

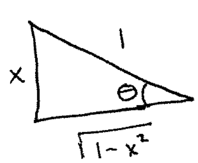
$du = 4\sec^2 4x\,dx$

$$= \frac{1}{4}\left(\tan 4x + \frac{2}{3}\tan^3 4x + \frac{1}{5}\tan^5 4x \right) + C$$

H. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1-x^2}{(1-x)^2}}\,dx = \int \frac{\sqrt{1-x^2}}{(1-x)}\,dx = \int \frac{\cos^2\theta\,d\theta}{1-\sin\theta} = \int \frac{1-\sin^2\theta}{1-\sin\theta}\,d\theta$$

Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$



$$= \int (1 + \sin\theta)\,d\theta$$

$$= \theta - \cos\theta + C$$

$$= \arcsin x - \sqrt{1-x^2} + C$$

12. i $\int \cos^4 2x \, dx = \frac{1}{4}\int (1+\cos 4x)^2 dx = \frac{1}{4}\int (1 + 2\cos 4x + \cos^2 4x)\, dx$

$= \frac{1}{4}\int \left(1 + 2\cos 4x + \frac{1}{2} + \frac{1}{2}\cos 8x\right) dx = \frac{1}{4}\left(\frac{3}{2}x + \frac{1}{2}\sin 4x + \frac{1}{16}\sin 8x\right) + C$

j. $\int \frac{5x^2 - 3x + 5}{x^3 + 5x}\, dx = \int \left(\frac{1}{x} + \frac{4x}{x^2+5} - \frac{3}{x^2+5}\right) dx$

$\frac{5x^2 - 3x + 5}{x(x^2+5)} = \frac{A}{x} + \frac{Bx + C}{x^2+5}$ $\qquad = \ln|x| + 2\ln(x^2+5) - \frac{3}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) + C$

$5x^2 - 3x + 5 = A(x^2+5) + (Bx + C)x$

Let $x = 0$: $5 = A(5) \Rightarrow A = 1$

Equate Coeff : x^2: $5 = A + B \Rightarrow B = 4$

x : $-3 = C$

k. $\int \frac{\sqrt{x}}{x-4}\, dx = \int \frac{u}{u^2-4} \cdot 2u\, du = \int \left(\frac{2u^2 - 8}{u^2-4} + \frac{8}{u^2-4}\right) du = \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right) du$

Let $u = \sqrt{x}$ $\qquad \frac{8}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$ $\qquad = 2\sqrt{x} + 2\ln|\sqrt{x}-2| - 2\ln|\sqrt{x}+2| + C$

so $u^2 = x$

$2u\,du = dx$ $\qquad 8 = A(u+2) + B(u-2)$

$u = 2$: $8 = 4A \Rightarrow A = 2$

$u = -2$: $8 = -4B \Rightarrow B = -2$

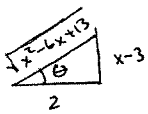
l. $\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int (1 - \cos^2 2x)\, dx = \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$

$= \frac{1}{4}\left(\frac{x}{2} - \frac{1}{8}\sin 4x\right) + C$

12. M. $\displaystyle\int \frac{2x}{(x^2-6x+13)^2}\,dx = \underbrace{\int \frac{2x-6}{(x^2-6x+13)^2}\,dx}_{(1)} + \underbrace{\int \frac{6}{((x-3)^2+2^2)^2}\,dx}_{(2)}$

For (1) use the u-sub $u = x^2-6x+13$, $du = (2x-6)\,dx$ to get $\displaystyle -\frac{1}{x^2-6x+13} + C$

For (2) let $(x-3) = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$ to get $\displaystyle\int \frac{12\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{3}{4}\int \cos^2\theta\,d\theta$



$$= \frac{3}{8}\left(\theta + \sin\theta\cos\theta\right) + C = \frac{3}{8}\left(\arctan\left(\frac{x-3}{2}\right) + \frac{2(x-3)}{x^2-6x+13}\right) + C$$

So $\displaystyle\int \frac{2x}{(x^2-6x+13)^2}\,dx = \frac{3}{8}\left(\arctan\left(\frac{x-3}{2}\right) + \frac{2x-6}{x^2-6x+13}\right) - \frac{1}{x^2-6x+13} + C$

N. $\displaystyle\int_{-1}^{4} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{4} \frac{dx}{x^2}$ both of which are divergent p-integrals, so the integral ~~can~~ diverges.

O. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt[3]{x}} = \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{4} x^{-1/3}\,dx = \lim_{R\to 0^-}\left(\frac{3}{2}R^{2/3} - \frac{3}{2}(-1)^{2/3}\right) + \lim_{S\to 0^+}\left(\frac{3}{2}4^{2/3} - \frac{3}{2}S^{2/3}\right)$

$$= \frac{3}{2}\left(4^{2/3} - 1\right)$$

P. $\displaystyle\int \frac{3x^2+11x+16}{(x+1)(x^2+6x+13)}\,dx = \int \left(\frac{1}{x+1} + \frac{2x}{x^2+6x+13} + \frac{3}{(x+3)^2+2^2}\right)dx$

$\displaystyle\frac{3x^2+11x+16}{(x+1)(x^2+6x+13)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+6x+13}$

$\displaystyle = \ln|x+1| + \ln\left(x^2+6x+13\right) + \frac{3}{2}\arctan\left(\frac{x+3}{2}\right) + C$

$3x^2+11x+16 = A(x^2+6x+13) + (Bx+C)(x+1)$

Let $x=-1$: $8 = A(8) \Rightarrow A = 1$

x^2: $3 = A+B \Rightarrow B = 2$

x^0: $16 = 13A + C \Rightarrow C = 3$

12. Q. $\displaystyle\int \frac{e^x}{e^{2x}-e^x}\,dx = \int \frac{e^{-x}}{1-e^{-x}}\,dx = \ln\left|1-e^{-x}\right| + C$

$u = 1-e^{-x}$

$du = e^{-x}\,dx$

Note: multiplying numerator & denominator by e^{-2x} is a trick I learned from Adrian Soto. Much easier than $u=e^x$ & using partial fractions.

R. $\displaystyle\int \frac{e^{2x}}{e^{2x}-e^x}\,dx = \int \frac{e^x}{e^x-1}\,dx = \ln\left|e^x-1\right| + C$

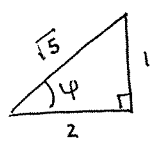
$u = e^x - 1$

$du = e^x\,dx$

S. $\displaystyle\int_0^1 \sqrt{4+x^2}\,dx = \int_0^{\varphi} 2\sec\theta \cdot 2\sec^2\theta\,d\theta = 2\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^{\varphi}$

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$



$= 2\left(\frac{\sqrt{5}}{2}\cdot\frac{1}{2} + \ln\left|\frac{\sqrt{5}}{2}+\frac{1}{2}\right|\right)$

$x=0 \longmapsto \theta = 0$

$x=1 \longmapsto \underbrace{\theta = \arctan\left(\tfrac{1}{2}\right)}_{=\varphi}$

T. $\displaystyle\int \sec x\tan^5 x\,dx = \int \left(\sec^2 x - 1\right)^2 \sec x\tan x\,dx = \int \left(u^4 - 2u^2 + 1\right)du$

$u = \sec x$

$du = \sec x\tan x\,dx$

$= \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$

U. $\displaystyle\int \frac{2x^2+11x+11}{(x+1)(x+2)(x+3)}\,dx = \int \left(\frac{1}{x+1} + \frac{3}{x+2} - \frac{2}{x+3}\right)dx = \ln|x+1| + 3\ln|x+2| - 2\ln|x+3| + C$

$$\frac{2x^2+11x+11}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$2x^2+11x+11 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Let $x=-1$: $2 = A(2) \Rightarrow A = 1$

$x=-2$: $-3 = B(-1) \Rightarrow B = 3$

Equate coeff

x^2: $2 = A + B + C$

$\Rightarrow C = -2$

13. A. $\displaystyle\int_0^\infty \frac{dx}{x^2+\sqrt{x}} = \underbrace{\int_0^1 \frac{dx}{x^2+\sqrt{x}}}_{(1)} + \underbrace{\int_1^\infty \frac{dx}{x^2+\sqrt{x}}}_{(2)}$

we consider the "tall" part first, (1).

Since $0 < \frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}$ for $x \in (0,1]$ & $\int_0^1 \frac{dx}{\sqrt{x}}$ is a convergent p-integral ($p = \frac{1}{2} < 1$),

By comparison, $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$ also converges.

Now consider the "long" part, (2).

Since $0 < \frac{1}{x^2+\sqrt{x}} < \frac{1}{x^2}$ for $x \in [1,\infty)$ & $\int_1^\infty \frac{dx}{x^2}$ is a convergent p-integral ($p = 2 > 1$),

by comparison, $\int_1^\infty \frac{dx}{x^2+\sqrt{x}}$ also converges.

Since (1) & (2) both converge, $\displaystyle\int_0^\infty \frac{dx}{x^2+\sqrt{x}}$ also converges.

B. $\displaystyle\int_\pi^\infty \frac{|\sin x|}{x^2+1}\,dx$ converges by comparison since $0 < \frac{|\sin x|}{x^2+1} < \frac{1}{x^2}$ for $x \geq \pi$

& $\int_\pi^\infty \frac{dx}{x^2}$ is a convergent p-integral ($p = 2 > 1$).

C. $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx = \underbrace{\int_{-\infty}^{-1} e^{-x^2}\,dx}_{(1)} + \underbrace{\int_{-1}^{1} e^{-x^2}\,dx}_{(2)} + \underbrace{\int_1^\infty e^{-x^2}\,dx}_{(3)}$

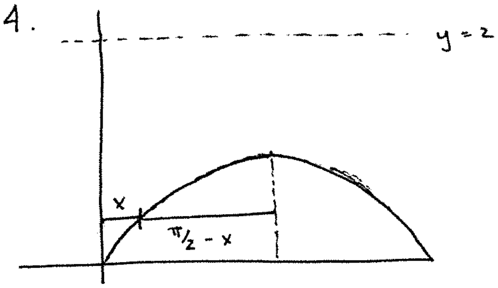
Since e^{-x^2} is continuous on $[-1,1]$, integral (2) is a proper integral & converges.

By symmetry (1) & (3) either both converge or both diverge, we consider only (3).

Since $0 < \frac{1}{e^{x^2}} < \frac{1}{x^2}$ & $\int_1^\infty \frac{dx}{x^2}$ is a convergent p-integral ($p=2>1$),

by comparison, integral (3) also converges. It follows that (1) does as well.

Since (1), (2) & (3) all converge, $\int_{-\infty}^\infty e^{-x^2}\,dx$ converges. In calc iii you will show $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.

14. 

The width at height y is $2\left(\frac{\pi}{2}-x\right) = \pi - 2\arcsin y$

so $\text{Force} = \rho g \int_0^1 (2-y)\left(\pi - 2\arcsin y\right)\,dy$

$u = \pi - 2\arcsin y \qquad dv = (2-y)\,dy$

$du = \frac{-2}{\sqrt{1-y^2}}\,dy \qquad v = 2y - \frac{y^2}{2}$

$$= \rho g\left[\left(\pi - 2\arcsin y\right)\left(2y - \frac{y^2}{2}\right)\Big|_0^1 + \int_0^1 \frac{4y - y^2}{\sqrt{1-y^2}}\,dy\right]$$

$y = \sin\theta \qquad 0 \longmapsto 0$

$dy = \cos\theta\,d\theta \qquad 1 \longmapsto \pi/2$

$$= \rho g\left[\left(\cancel{\pi - 2\cdot\tfrac{\pi}{2}}^{\,0}\right)\left(2-\tfrac{1}{2}\right) - (\pi)(0)\right.$$

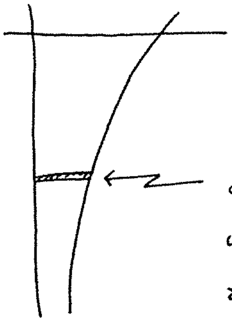
$$\left. + \int_0^{\pi/2} \frac{4\sin\theta - \sin^2\theta}{\cancel{\cos\theta}}\,\cancel{\cos\theta}\,d\theta\right]$$

$$= \rho g\left(-4\cos\theta - \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right)\Big|_0^{\pi/2}$$

$$= \rho g\left(4 - \frac{1}{2}\left(\frac{\pi}{2}\right)\right)$$

$$= \rho g\left(4 - \frac{\pi}{4}\right)$$

15.



depth $= -y$

width $= x = e^y$

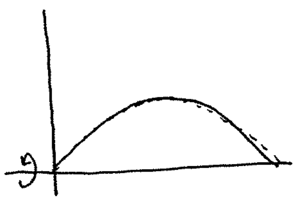
area $= e^y \Delta y$

$$\text{Force} = \rho g \int_{-\infty}^{0} -y e^y \, dy = \rho g (-1) \int_{-\infty}^{0} y e^y \, dy$$

see #4.A.

$$= \rho g (-1)(-1) = \rho g$$

16.



$y = \sin x$

$1 + (y')^2 = 1 + \cos^2 x$

$$S = 2\pi \int_0^{\pi} \sin x \sqrt{1+\cos^2 x} \, dx$$

see #12.E.

$$= 2\pi \left(-\frac{1}{2}\right)\left(\cos x \sqrt{1+\cos^2 x} + \ln\left| \sqrt{1+\cos^2 x} + \cos x \right| \right)\Bigg|_0^{\pi}$$

$$= -\pi \left[-1\sqrt{2} + \ln\left|\sqrt{2} - 1\right| - 1\sqrt{2} - \ln\left|\sqrt{2}+1\right| \right]$$

$$= \pi \left(2\sqrt{2} - \ln(\sqrt{2}-1) + \ln(\sqrt{2}+1) \right)$$