Exam 3 Review

1\. A. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ $\qquad \displaystyle\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} \Rightarrow 2 = A(n+1) + B(n-1)$

$n = 1, \quad 2 = 2A \quad$ so $A = 1$

$n = -1, \quad 2 = -2B \quad$ so $B = -1$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$S_n = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}\right) + \cdots + \left(\cancel{\tfrac{1}{n-2}} - \tfrac{1}{n}\right) + \left(\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n+1}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$S = \lim_{n \to \infty} S_n = \frac{3}{2}$$

B. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-n}$ $\qquad \displaystyle\frac{2}{n(n-1)} = \frac{A}{n-1} + \frac{B}{n} \Rightarrow 2 = An + B(n-1)$

$n = 0, \quad 2 = -B \quad$ so $B = -2$

$n = 1, \quad 2 = A \quad$ so $A = 2$

$$= \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n} \right)$$

$$S_n = (2 - \cancel{1}) + \left(\cancel{1} - \cancel{\tfrac{2}{3}}\right) + \left(\cancel{\tfrac{2}{3}} - \cancel{\tfrac{2}{4}}\right) + \cdots + \left(\cancel{\tfrac{2}{n-1}} - \tfrac{2}{n}\right)$$

$$= 2 - \frac{2}{n}$$

$$S = \lim_{n \to \infty} S_n = 2$$

C. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n}} = \sum_{n=1}^{\infty} 2^2 \left(\frac{2}{9}\right)^n = \frac{8/9}{1 - 2/9} = \frac{8}{9} \cdot \frac{9}{7} = \frac{8}{7}$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{4}{3}\right)^n$ diverges since $|r| = 4/3 \geq 1$.

1.e. $\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \cdots = \frac{1/9}{1 - 8/9} = 1$ in general, $\frac{\text{first term}}{1 - \text{ratio}}$ when $|\text{ratio}| < 1$

f. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$ diverges since $|r| = \frac{3}{2} > 1$.

g. $(\ln 1 - \ln \frac{1}{2}) + (\ln \frac{1}{2} - \ln \frac{1}{3}) + (\ln \frac{1}{3} - \ln \frac{1}{4}) + \cdots$

Let $S_n = (\ln 1 - \cancel{\ln \frac{1}{2}}) + (\cancel{\ln \frac{1}{2}} - \cancel{\ln \frac{1}{3}}) + \cdots + (\cancel{\ln \frac{1}{n}} - \ln \frac{1}{n-1})$

$= \ln 1 - \ln \frac{1}{n-1} = 0 + \ln(n-1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n-1) = \infty.$

2. A. $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Let $a_n = \frac{1}{n^2-1}$ & $b_n = \frac{1}{n^2}$. Since $a_n, b_n > 0$ we can apply the Limit Comparison Test. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ both series behave the same. We know $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$, so $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ also converges by the Limit Comparison Test.

B. $\sum_{n=1}^{\infty} \frac{\sqrt{9n+3}}{n+2}$

<u>LCT solution</u>: Let $a_n = \frac{\sqrt{9n+3}}{n+2}$ & $b_n = \frac{1}{\sqrt{n}}$. Since $a_n, b_n > 0$ & $\lim_{n\to\infty} \frac{a_n}{b_n} = 3$, the Limit Comparison Test implies that $\sum_{n=1}^{\infty} \frac{\sqrt{9n+3}}{n+2}$ diverges since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $(p = 1/2 \leq 1)$.

<u>Direct Comparison solution</u>:

Initially we note $0 < \frac{1}{\sqrt{n}} < \frac{3\sqrt{n}}{2n} = \frac{\sqrt{9n}}{n+n} < \frac{\sqrt{9n+3}}{n+2}$ for $n > 2$. Since $\sum \frac{1}{\sqrt{n}}$ is a divergent p-series $(p = 1/2 \leq 1)$, by the Comparison Test $\sum_{n=1}^{\infty} \frac{\sqrt{9n+3}}{n+2}$ also diverges.

2. c $\displaystyle\sum_{n=2}^{\infty} \frac{3n+7}{\sqrt[3]{n^5-1}}$

Since $0 < \dfrac{1}{n^{2/3}} = \dfrac{n}{\sqrt[3]{n^5}} < \dfrac{3n+7}{\sqrt[3]{n^5-1}}$ & $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{2/3}}$ is a divergent p-series $(p = 2/3 \le 1)$,

by comparison $\displaystyle\sum_{n=2}^{\infty} \frac{3n+7}{\sqrt[3]{n^5-1}}$ also diverges.

D. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n}+n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n}+n^2}$ converges by comparison to the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{n^2} = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}$ $(p = 2 > 1)$

since $0 < \dfrac{\pi}{\sqrt{n}+n^2} < \dfrac{\pi}{n^2}$.

E. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

LCT solution:

Let $a_n = \sin\left(\frac{1}{n^2}\right)$ & $b_n = \frac{1}{n^2}$. Since $a_n, b_n > 0$, we can apply the Limit Comparison Test.

To do so we compute $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{1/n^2} = \lim_{x\to\infty} \frac{\sin(1/x^2)}{1/x^2} \underset{\frac{0}{0}\text{ form}}{\overset{LH}{=}} \lim_{x\to\infty} \frac{\cos(1/x^2)\cdot \frac{-2}{x^3}}{-2/x^3}$

$$= \lim \left(\cos(1/x^2)\right) = \cos 0 = 1$$

Since the limit is positive & finite, $\sum a_n$ & $\sum b_n$ behave the same. In this case

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$, $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ also converges.

Direct Comparison solution:

Since $\sin x = x - \dfrac{x^3}{3!} + \ldots$ satisfies the A.S.T., for $x > 0$, $\sin x < x$. In particular,

$0 < \sin\left(\frac{1}{n^2}\right) < \frac{1}{n^2}$ & $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$. By comparison $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ also converges.

2. F. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{(n+1)^3}$

We know that for any $a > 0$, $\ln n < n^a$ for sufficiently large n.

With that in mind we make the comparison

$$0 \leq \frac{\ln n}{(n+1)^3} < \frac{n}{(n+1)^3} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$, by comparison

$\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{(n+1)^3}$ also converges.

3. A. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Let $f(x) = x e^{-x^2}$. $f(x)$ is positive & continuous. $f'(x) = e^{-x^2}(1 - 2x^2) < 0$

for $x > \frac{1}{\sqrt{2}}$, so $f(x)$ is decreasing and the Integral Test applies. We now consider

the integral

$$\int_1^{\infty} x e^{-x^2}\, dx = \frac{1}{2}\int_1^{\infty} e^{-u}\, du = \lim_{R\to\infty} -\frac{1}{2}e^{-u}\Big|_1^R = \lim_{R\to\infty}\left(-\frac{1}{2}e^{-R} + \frac{1}{2e}\right) = \frac{1}{2e}$$

Let $u = x^2$

$du = 2x\, dx$

Since the integral converges, by the Integral Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ also converges.

B. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{\ln n}}$

Let $f(x) = \dfrac{1}{x\sqrt[3]{\ln x}} = (x\sqrt[3]{\ln x})^{-1}$. Since $f'(x) = -(x\sqrt[3]{\ln x})^{-2}\left(\sqrt[3]{\ln x} + \frac{1}{3}(\ln x)^{-2/3}\right) < 0$ for $x > 1$,

f is positive, decreasing & continuous for $x > 1$. The Integral Test applies, so we consider

$$\int_2^{\infty} \frac{dx}{x(\ln x)^{1/3}} = \int_{\ln 2}^{\infty} \frac{du}{u^{1/3}}$$ which is a divergent p-integral $(p = 1/3 \leq 1)$. Since the integral diverges,

by the Integral Test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{\ln n}}$ also diverges.

$u = \ln x$

$du = \frac{1}{x}dx$

4\. A. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n \cos 5n}{n^2+3}$

Since $0 < \left| \dfrac{\sin n \cos 5n}{n^2+3} \right| < \dfrac{1}{n^2}$ & $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$,

by comparison $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n \cos 5n}{n^2+3} \right|$ also converges. We conclude then that

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin n \cos 5n}{n^2+3}$ converges absolutely.

B. $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$ Note: $\cos(\pi n) = (-1)^n$

Since $0 < \dfrac{1}{n} < \left| \dfrac{\cos(\pi n)}{n-1} \right|$ & $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent Harmonic series, by comparison

$\displaystyle\sum_{n=2}^{\infty} \left| \frac{\cos(\pi n)}{n-1} \right|$ also diverges. So, $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$ does not converge absolutely.

To test for conditional convergence we use the Alternating Series Test.

Since $a_n = \dfrac{1}{n-1} > 0$, $a_n = \dfrac{1}{n-1} > \dfrac{1}{n} = a_{n+1}$ & $\displaystyle\lim_{n \to \infty} \frac{1}{n-1} = 0$, $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$

converges by the Alternating Series Test.

Since $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$ converges, but not absolutely, it converges conditionally.

C. $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{n^2-2n}$

We begin by testing for absolute convergence, i.e. we consider the series $\displaystyle\sum_{n=3}^{\infty} \left| \frac{(-1)^{n-1}}{n^2-2n} \right| = \sum_{n=3}^{\infty} \frac{1}{n^2-2n} = \sum_{n=3}^{\infty} a_n$

We use the Limit Comparison Test to compare to $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2} = \sum_{n=3}^{\infty} b_n$. Since $\displaystyle\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ & $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a

convergent p-series $(p = 2 > 1)$, both positive series converge.

Since $\displaystyle\sum_{n=3}^{\infty} \left| \frac{(-1)^{n-1}}{n^2-2n} \right|$ converges, by definition $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{n^2-2n}$ converges absolutely.

4. D. $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n (n-1)}{(n+1)}$

Since $\displaystyle\lim_{n\to\infty} \frac{(-1)^n (n-1)}{(n+1)}$ does not exist, by the Test for Divergence $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n (n-1)}{n+1}$ diverges.

E. $\displaystyle\sum_{n=5}^{\infty} \left(\frac{2n+3}{5n-7}\right)^n$

We apply the root test to see

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{5n-7}\right)^n\right|} = \lim_{n\to\infty} \left(\frac{2n+3}{5n-7}\right) = \frac{2}{5} < 1.$$

Since $L < 1$, by the Root Test $\displaystyle\sum_{n=5}^{\infty} \left(\frac{2n+3}{5n-7}\right)^n$ converges absolutely.

F. $\displaystyle\sum_{n=6}^{\infty} \frac{2^n n^2}{(2n+5)!}$

We apply the Ratio Test.

$$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1} (n+1)^2}{(2n+7)!} \cdot \frac{(2n+5)!}{2^n \, n^2} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{(2n+5)!}{(2n+7)(2n+6)(2n+5)!}$$

$$= \lim_{n\to\infty} 2 \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{(2n+7)(2n+6)} = 0$$

Since $\rho < 1$, by the Ratio Test $\displaystyle\sum_{n=6}^{\infty} \frac{2^n n^2}{(2n+5)!}$ converges absolutely.

5. $\displaystyle\cos\pi = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = 1 - \frac{\pi^2}{2} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$ is an alternating series

that satisfies the assumptions of the Alternating Series Test. We can apply

the error bound $|S - S_n| < a_{n+1}$, in particular

$$\left| \cos\pi - \left(1 - \frac{\pi^2}{2} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!}\right) \right| < \frac{\pi^{10}}{10!}$$

6\. A. $\sum_{n=0}^{\infty} \frac{n^n}{n!}(x-e)^n$

Applying the Ratio Test gives

$$\rho = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}(x-e)^{n+1} \Big/ \frac{n^n}{(n)!}(x-e)^n\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)n!}\cdot\left|\frac{(x-e)^{n+1}}{(x-e)^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}\cdot\frac{\cancel{(n+1)}}{\cancel{(n+1)}}\cdot|x-e| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n|x-e| = \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}|x-e|$$

Note: $\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^c$

$$= e\,|x-e|$$

The series converges when $\rho < 1$, i.e. $e\,|x-e| < 1$

or $|x-e| < \frac{1}{e}$, so $R = \frac{1}{e}$

B. $\sum_{n=1}^{\infty}(2n-1)!\,(x+2)^n$

Applying the Ratio Test gives

$$\rho = \lim_{n\to\infty}\left|\frac{(2n+1)!\,(x+2)^{n+1}}{(2n-1)!\,(x+2)^n}\right| = \lim_{n\to\infty}(2n+1)(2n)\,|x+2| = \begin{cases}\infty & \text{if } x \neq -2\\ 0 & \text{if } x = -2\end{cases}$$

The series converges when $\rho < 1$, i.e. only for $x = -2$, so $R = 0$.

C. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$

One more time, we apply the Ratio Test to find

$$\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty}\frac{|x^2|}{(2n+3)(2n+2)} = 0$$

Since $\rho = 0 < 1$ for all x, $R = \infty$

7\. A. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

To find R we use the Root Test.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n}\right|} = |x| \qquad \text{Note: } \sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$$

So $|x| < 1$

Checking the end points gives

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the <u>convergent</u> Alternating Harmonic Series

$x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the <u>divergent</u> Harmonic Series.

So the interval of convergence is $I = [-1, 1)$.

B. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(x+4)^{2n}$

To find R we use the Root Test.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{4^n}(x+4)^{2n}\right|} = \lim_{n\to\infty} \frac{(x+4)^2}{4} = \frac{(x+4)^2}{4}$$

The series converges when $L < 1$, i.e. $\frac{(x+4)^2}{4} < 1$ so $(x+4)^2 < 4$

$|x+4| < 2$

Checking the endpoints gives

$x = -2$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(2)^{2n} = \sum_{n=0}^{\infty} (-1)^n$ which diverges by the Test for Divergence

$x = -6$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(-2)^{2n} = \sum_{n=0}^{\infty} (-1)^n$ ———— " ————

So the interval of convergence is $I = (-6, -2)$.

7.c $\sum_{n=1}^{\infty} \frac{n-1}{n2^n}(x-2)^n$

To find R we use the Root Test.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n-1}{n2^n}(x-2)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{n-1}{n}} \cdot \frac{|x-2|}{2} = \frac{|x-2|}{2}$$

The series converges when $\frac{|x-2|}{2} < 1$, i.e. $|x-2| < 2$.

Checking the end points gives

$x = 0$: $\sum_{n=1}^{\infty} \frac{n-1}{n2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{n-1}{n}(-1)^n$ which diverges by the Test for Divergence.

$x = 4$: $\sum_{n=1}^{\infty} \frac{n-1}{n2^n}(2)^n = \sum_{n=1}^{\infty} \frac{n-1}{n}$ which also diverges by the Test for Divergence.

So, the interval of convergence is $I = (0, 4)$

D. $\sum_{n=0}^{\infty} n!\,(x+\pi)^n$

To find R we use the Ratio Test.

$$\rho = \lim_{n\to\infty} \left|\frac{(n+1)!\,(x+\pi)^{n+1}}{n!\,(x+\pi)^n}\right| = \lim_{n\to\infty} (n+1)\,|x+\pi| = \begin{cases} \infty & \text{if } x \neq -\pi \\ 0 & \text{if } x = -\pi \end{cases}$$

The series only converges when $x = -\pi$, i.e. the interval of convergence is the single point $\{-\pi\}$.

E. $I = (-\infty, \infty)$, see 6.c.

F. $\sum_{n=1}^{\infty} \frac{(6x-12)^n}{n^2}$

To find R we again apply Root Test $L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(6x-12)^n}{n^2}\right|} = |6x-12|$ so the series converges for $|x-2| < \frac{1}{6}$. For $x = \frac{11}{6}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$x = \frac{13}{6}$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. So $I = [11/6, 13/6]$

8. A $f(x) = \dfrac{4x}{2+x} = \dfrac{2x}{1-(-\frac{x}{2})} = 2x \cdot \dfrac{1}{1-(-\frac{x}{2})}$ so

$$f(x) = 2x \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{n-1}} \quad \text{for} \quad \left|-\frac{x}{2}\right| < 1, \text{ i.e. } |x| < 2$$

B. $g(x) = \dfrac{\sin x}{x} = \dfrac{1}{x} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$ for all x

C. $h(x) = x e^{x^2} = x \displaystyle\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ for all x

D. $k(x) = \dfrac{5}{1+x^2} = 5 \dfrac{1}{1-(-x^2)} = 5 \displaystyle\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} 5(-1)^n x^{2n}$ for $|-x^2| < 1$

i.e. $|x| < 1$

9. A. $f(x) = (5-x)^{3/2}$; $c = 1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(5-x)^{3/2}$	8
1	$\frac{3}{2}(5-x)^{1/2}(-1)$	-3
2	$\frac{3}{2}\cdot\frac{1}{2}(5-x)^{-1/2}$	$3/8$
3	$\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{-1}{2}(5-x)^{-3/2}(-1)$	$3/64$

so $f(x) = 8 - 3(x-1) + \frac{3}{16}(x-1)^2 + \frac{1}{128}(x-1)^3 + \cdots$

9.B. $g(x) = \sin 2x \;;\; c = \pi/4$

n	$g^{(n)}(x)$	$g^{(n)}(\pi/4)$
0	$\sin 2x$	1
1	$2\cos 2x$	0
2	$-4\sin 2x$	-4
3	$-8\cos 2x$	0
	$\vdots$	$\vdots$

So $g(x) = 1 - \frac{4}{2!}(x-\pi/4)^2 + \frac{16}{4!}(x-\pi/4)^4 - \frac{64}{6!}(x-\pi/4)^6 + \cdots$

10. A. $$\lim_{x\to 0} \frac{\sin x - x + x^3/6}{2x^5} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \cdots\right) - x + x^3/6}{2x^5}$$

$$= \lim_{x\to 0} \frac{\frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots}{2x^5}$$

$$= \lim_{x\to 0} \left(\frac{1}{240} - \frac{x^2}{2\cdot 7!} + \frac{x^4}{2\cdot 9!} - \cdots\right) = \frac{1}{240}$$

B. $$\lim_{x\to 0} \frac{x\cos x}{1-e^x} = \lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}{1 - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)} = \lim_{x\to 0} \frac{x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots}{-x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots}$$

$$= \lim_{x\to 0} \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}{-1 - \frac{x}{2!} - \frac{x^2}{3!} - \cdots} = -1$$

11. A.

$$f(x) = \sin 2x + e^{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$= \left(2x - \frac{8}{3!}x^3 + \cdots \right) + \left(1 + x^2 + \frac{x^4}{2!} + \cdots \right)$$

$$= 1 + 2x + x^2 - \frac{8}{3}x^3 + \cdots$$

B. $$g(x) = \frac{e^x}{1-x} = e^x \cdot \frac{1}{1-x} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \right)\left(1 + x + x^2 + x^3 + \cdots \right)$$

$$= 1 + 2x + \left(x^2 + x^2 + \frac{x^2}{2} \right) + \left(x^3 + x^3 + \frac{x^3}{2} + \frac{x^3}{6} \right) + \cdots$$

$$= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \cdots$$

12. Choose all series that can be <u>shown</u> to diverge using the Divergence Test.

(A) $\sum \frac{(-1)^n n}{3-n}$ Since $\lim_{n\to\infty} \frac{(-1)^n n}{3-n}$ does not exist, this series diverges by the Divergence Test.

(B) $\sum \frac{2}{n}$ Although this series diverges, $\lim_{n\to\infty} \frac{2}{n} = 0$, so the Divergence Test does not apply.

(C) $\sum \frac{1}{n^4+1}$ This series <u>converges</u>.

13. True/False.

(a) **T** / **F**: If $\lim_{n\to\infty} a_n = 7$, the series $\sum a_n$ converges.

(b) **T** / **F**: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ converges.

(c) **T** / **F**: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ diverges.

$\lim_{n\to\infty} a_n = 0$ gives no information

(d) **T** / **F**: If $0 < a_n < \frac{1}{n}$, the series $\sum a_n$ diverges. $0 < \frac{1}{n^2} < \frac{1}{n}$, $\sum \frac{1}{n^2}$ converges

(e) **T** / **F**: The Harmonic series $\sum \frac{1}{n}$ converges.

(f) **T** / **F** : The Alternating Harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges. Follows from A.S.T.

(g) **T** / **F**: If $\sum |a_n|$ converges, then $\sum a_n$ converges conditionally.

(h) **T** / **F**: If $\sum |a_n|$ diverges, then $\sum a_n$ diverges. see (f)

$\sum |a_n|$ converges is the definition of $\sum a_n$ converging absolutely

(i) **T** / **F** : $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = 0$ $\quad \sin(\pi) = 0$

(j) **T** / **F**: $\sum \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum \frac{1}{n} - \sum \frac{1}{n+1}$

Linearity requires the series to converge.

* No work is expected for this type of problem.

14. For each of the following series determine if they converge or diverge and then choose a test that can be used to show that.

(a) $\sum \left(1+\frac{1}{n}\right)^{-n^2}$ Converges / Diverges by the Root Test / Divergence Test.

$$\sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{-n^2}\right|} = \left(1+\frac{1}{n}\right)^{-n} \xrightarrow[n\to\infty]{} \frac{1}{e}$$

(b) $\sum \left(1+\frac{1}{n}\right)^{n}$ Converges / Diverges by the Root Test / Divergence Test.

$$\left(1+\frac{1}{n}\right)^{n} \xrightarrow[n\to\infty]{} e \neq 0$$

(c) $\sum \frac{1}{n \ln n}$ Converges / Diverges by the Ratio Test / Integral Test.

Similar to 3 (b)

(d) $\sum \frac{n+1}{n^2+2}$ Converges / Diverges by the Limit Comparison Test / Ratio Test.

Compare to $\sum \frac{1}{n}$

(e) $\sum \frac{(-1)^n}{n}$ Converges / Diverges by the Integral Test / Alternating Series Test.

(f) $\sum (-n)^n$ Converges / Diverges by the Divergence Test / Alternating Series Test.

15. Use a series to integrate.

(A) $$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{so} \quad \int e^{x^2}\,dx = A + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,n!}$$

constant.

(B) $$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \quad \text{so} \quad \int \frac{\sin x}{x}\,dx = A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

(C) $$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \quad \text{so} \quad \int \sin x^2\,dx = A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

16. None! The Divergence Test can never be used to show convergence.

17. Only (C). (A) requires the Integral Test, (B) is not positive.

18. Both (A) & (B). (C) diverges by the Divergence Test

19. None! The A.S.T. can not be used to show divergence.

20. Both (B) & (C). Rational functions like (A) are always inconclusive.

21. Both (B) & (C).

16

22. Match a series on the left with a statement on the right to make a true statement.

$1 - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{10} - \frac{1}{27} + \frac{1}{28} - \cdots$ C

Note: C also applies

$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ E

if this was n=0, then F

$\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}\right)$ G

if this was n, then H

$\sum_{n=7}^{\infty} 13^{3-2n} = \sum_{n=7}^{\infty} 13^3\left(\frac{1}{13^2}\right)^n$ I

$\sum_{n=1}^{\infty} n^n (x-7)^n$ B

a) has radius of convergence $R = 7$.

b) has radius of convergence $R = 0$.

c) is a convergent alternating series.

d) is a divergent alternating series.

e) is the Maclaurin series for $\cos x - 1$.

f) is the Maclaurin series for $\cos x$.

g) is the Maclaurin series for $\sin x$.

h) is the Maclaurin series for $-\sin x$.

i) is a convergent geometric series.

j) is a divergent geometric series.