

Zeno - greek philosopher ~490-430 BC

Dichotomy paradox:

To travel a distance 1, first must travel  $\frac{1}{2}$ , then half of what is left, i.e.  $\frac{1}{4}$ , then half of what is left, i.e.  $\frac{1}{8}$ , etc.

Since the sequence is infinite, the distance cannot be travelled

The steps are terms in a sequence

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \right\}$$

the general term is given by  $a_n = a(n) = \frac{1}{2^n}$  for  $n \in \{1, 2, 3, \dots\}$

Other equivalent notation

$$\left\{ \frac{1}{2^n} \right\}, \left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty}$$

Note:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \quad \leftarrow \text{This is an infinite } \underline{\text{series}}$$

$\therefore$  will be the topic of

§10.2  $\in$  the rest of Ch. 10.

## Examples of sequences

$$\{2, 4, 6, 8, \dots\} = \left\{2n\right\}_{n=1}^{\infty} = a_n = 2n$$

$$\left\{\frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots\right\} = \left\{\left(\frac{2}{3}\right)^n\right\}_{n=2}^{\infty} = b_n = \left(\frac{2}{3}\right)^n, n=2$$

$$\{-1, 1, -1, 1, \dots\} \leftarrow \left\{\cos(\pi n)\right\} = c_n = \cos(\pi n)$$

↑ if  $n$  not specified, assume 1, but probably not important

Sequences can be defined recursively.

## Fibonacci, 1202

- in month 1, we have 1 pair of immature rabbits
- rabbits mature in one month
- a mature pair produced a new immature pair each month
- rabbits never die

Month	Immature	Mature	Total	$\begin{cases} F_0 = 0 \\ F_1 = 1 \end{cases}$
1	1	0	1	
2	0	1	1	$F_2 = F_1 + F_0 = 1$
3	1	1	2	⋮
4	1	2	3	$F_n = F_{n-1} + F_{n-2}$
5	2	3	5	This is a recursive sequence
				$\{0, 1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

Definition: We say  $\{z_n\}$  converges to  $L$

and write  $\lim_{n \rightarrow \infty} z_n = L$  or  $z_n \rightarrow L$  as  $n \rightarrow \infty$   
 or  $z_n \xrightarrow{n \rightarrow \infty} L$

if, for every  $\epsilon > 0$ , there exists  $M$  such that

$$|z_n - L| < \epsilon \text{ when } n > M.$$

- If no limit exists, we say the sequence diverges.
- If  $L = \infty$ , we say the sequence diverges to infinity.

$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\} = \left\{ \frac{1}{2^n} \right\}$  converges to 0

$$\text{OR } \frac{1}{2^n} \xrightarrow{n \rightarrow \infty} 0$$

$\{0, 1, 1, 2, 3, 5, \dots\} = \{f_n\}$  diverges to  $\infty$  (since rabbits never die!)

$$\text{OR } f_n \xrightarrow{n \rightarrow \infty} \infty$$

$\{-1, 1, -1, 1, -1, 1, \dots\} = \{\cos(\pi n)\} = \{(-1)^n\}$  diverges since it oscillates

Note: formally, sequences are functions with domain

$$\mathbb{N} = \{1, 2, 3, \dots\} \text{ the natural numbers}$$

or some subset of the integers

$$\{-2, -1, 0, 1, 2, \dots\} \text{ or } \{2, 3, 4, \dots\} \text{ or etc.}$$

Often we can use what we know about limits of continuous functions  
 on  $\mathbb{R}$ .

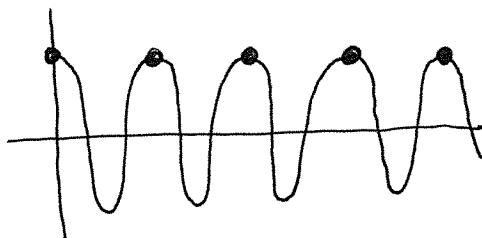
Theorem: If  $\lim_{x \rightarrow \infty} f(x) = L$  exists, then  $\lim_{n \rightarrow \infty} f(n) = L$

Note:  $\lim_{n \rightarrow \infty} f(n) = L \not\Rightarrow \lim_{x \rightarrow \infty} f(x) = L$

example  $f(n) = \cos(2\pi n) = 1$  for all  $n \in \mathbb{N}$

$$\text{so } \lim_{n \rightarrow \infty} \cos(2\pi n) = 1$$

but  $\lim_{x \rightarrow \infty} \cos(2\pi x)$  does not exist.



$$\left\{ \frac{d_n}{n} \right\}_{n=1}^{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{d_n}{x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{1}{x} = 0, \text{ Since the limit exists,}$$

$$\lim_{n \rightarrow \infty} \frac{d_n}{n} = 0 \text{ as well.}$$

Note: Do not apply L'Hopital's Rule to terms of a sequence.

Sequences are not differentiable functions, not even continuous!

$$\left\{ \frac{3^n}{n!} \right\}$$

Note:  $n!$  is defined as: recursively  $0! = 1$   
 $n! = n(n-1)!$  for  $n > 1$

- or -

loosely/intuitively

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

$$\lim_{n \rightarrow \infty} \frac{3^n}{n!} = \lim_{n \rightarrow \infty} \left( \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdot \cdots \cdot \frac{3}{n} \right) ?$$

Limit Laws hold. if  $a_n \rightarrow A$  &  $b_n \rightarrow B$  as  $n \rightarrow \infty$

$$(a_n \pm b_n) \rightarrow A \pm B \text{ as } n \rightarrow \infty$$

$$\frac{a_n b_n}{a_n + b_n} \rightarrow AB \quad \text{as } n \rightarrow \infty$$

$$\left( \frac{a_n}{b_n} \right) \rightarrow \frac{A}{B} \quad \text{as } n \rightarrow \infty$$

$$c a_n \rightarrow cA \quad \text{as } n \rightarrow \infty$$

Squeeze Theorem applies

if  $a_n \leq b_n \leq c_n$  for  $n > M$ ,  $a_n \rightarrow L$  &  $c_n \rightarrow L$

then  $b_n \rightarrow L$  as well.

$$\text{Returning to } \lim \frac{3^n}{n!} \quad 0 \leq \frac{3^n}{n!} \leq \frac{9}{2} \cdot \frac{3}{n} \quad \begin{matrix} 0 \rightarrow 0 \\ \frac{9}{2} \cdot \frac{3}{n} \rightarrow 0 \end{matrix}$$

$$\text{so by Sq. Th. } \frac{3^n}{n!} \rightarrow 0 \text{ as } n \rightarrow \infty$$