Dichotomy paradox:

To travel a distance 1, first must travel $\frac{1}{2}$, then half of what is left, i.e. $\frac{1}{4}$, then half of what is left, i.e. $\frac{1}{8}$, etc.

Since the sequence is infinite, the distance cannot be travelled.

The steps are terms in a sequence:

\[
\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\}
\]

The general term is given by $a_n = a(2^n) = \frac{1}{2^n}$ for $n \in \{1, 2, 3, \ldots\}$

Other equivalent notation:

\[
\left\{ \frac{1}{2^n} \right\}, \left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty}
\]

Note:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1
\]

This is an infinite series, and will be the topic of §10.2 & the rest of Ch. 10.
Examples of sequences

\[
\{2, 4, 6, 8, \ldots \} = \{2^n\}_{n=1}^\infty = a_n = 2n
\]

\[
\{\frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots \} = \\{\left(\frac{2}{3}\right)^n\}_{n=2}^\infty = b_n = \left(\frac{2}{3}\right)^n, \quad n = 2
\]

\[
\{-1, 1, -1, 1, \ldots \} \leftarrow \{\cos(\pi n)\}_{n=1}^\infty = c_n = \cos(\pi n)
\]

if \(n\) not specified, assume 1, but probably not important

Sequences can be defined recursively.

\[
\text{Fibonacci}, \quad 1202
\]

- in month 1, we have 1 pair of immature rabbits
- rabbits mature in one month
- a mature pair produces a new immature pair each month
- rabbits never die

<table>
<thead>
<tr>
<th>Month</th>
<th>Immature</th>
<th>MATURE</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= F_1 + F_0 = 1 \\
F_3 &= F_2 + F_1 = 2 \\
F_4 &= F_3 + F_2 = 3 \\
F_5 &= F_4 + F_3 = 5 \\
\end{align*}
\]

This is a recursive sequence

\[
\{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \}
\]
Definition: We say \( \{a_n\} \) converges to \( L \)

and write \( \lim_{n \to \infty} a_n = L \) or \( a_n \to L \) as \( n \to \infty \)

or \( a_n \overset{n \to \infty}{\longrightarrow} L \)

if, for every \( \varepsilon > 0 \), there exists \( M \) such that

\[ |a_n - L| < \varepsilon \quad \text{when} \quad n > M. \]

- If no limit exists, we say the sequence diverges.
- If \( L = \infty \), we say the sequence diverges to infinity.

\( \{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \} = \{ \frac{1}{2^n} \} \) converges to 0

or \( \frac{1}{2^n} \overset{n \to \infty}{\longrightarrow} 0 \)

\( \{0, 1, 1, 2, 3, 5, \ldots \} = \{ F_n \} \) diverges to \( \infty \) (since rabbits never die!)

or \( F_n \overset{n \to \infty}{\longrightarrow} \infty \)

\( \{-1, 1, -1, 1, -1, 1, \ldots \} = \{ \cos (\pi n) \} = \{ (-1)^n \} \) diverges since it oscillates.

Note: Finally, sequences are functions with domain

\[ \mathbb{N} = \mathbb{N} - \{1, 2, 3, \ldots \} \] the natural numbers

or some subset of the integers

\( \{-2, -1, 0, 1, 2, \ldots \} \quad \text{or} \quad \{ 2, 3, 4, \ldots \} \quad \text{or etc.} \)

Often we can use what we know about limits of continuous functions on \( \mathbb{R} \).
Theorem: If \( \lim_{x \to \infty} f(x) = L \) exists, then \( \lim_{n \to \infty} f(n) = L \).

Note: \( \lim_{n \to \infty} f(n) = L \) \( \neq \) \( \lim_{x \to \infty} f(x) = L \).

Example: \( f(n) = \cos(2\pi n) = 1 \) for all \( n \in \mathbb{N} \).

So \( \lim_{n \to \infty} \cos(2\pi n) = 1 \).

But \( \lim_{x \to \infty} \cos(2\pi x) \) does not exist.

\[ \left\{ \frac{\ln n}{n} \right\}_{n=1}^{\infty} \]

\[ \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0 \quad \text{Since the limit exists,} \]

\[ \lim_{n \to \infty} \frac{\ln n}{n} = 0 \quad \text{as well.} \]

Note: Do not apply L'Hôpital's Rule to terms of a sequence.

Sequences are not differentiable functions, not even continuous!
\[
\left\{ \frac{3^n}{n!} \right\}
\]

Note: \( n! \) is defined recursively as:

\[
n! = n \cdot (n-1)! \quad \text{for } n \in \mathbb{N}
\]

\[
\lim_{n \to \infty} \frac{3^n}{n!} = \lim_{n \to \infty} \left( \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{n} \right)
\]

Limit Laws hold. If \( a_n \to A \) and \( b_n \to B \) as \( n \to \infty \):

\[
\left( a_n \pm b_n \right) \to A \pm B \quad \text{as } n \to \infty
\]

\[
a_n \cdot b_n \to A \cdot B
\]

\[
\left( \frac{a_n}{b_n} \right) \to \frac{A}{B}
\]

\[
C \cdot a_n \to C \cdot A \quad \text{as } n \to \infty
\]

Squeeze Theorem applies

If \( a_n \leq b_n \leq c_n \) for \( n \to \infty \) and \( a_n \to L \) and \( c_n \to L \),

then \( b_n \to L \) as well.

Returning to \( \lim_{n \to \infty} \frac{3^n}{n!} \);

\[
0 \leq \frac{3^n}{n!} \leq \frac{9 \cdot 3^n}{2^n} \quad \text{so } \frac{9 \cdot 3^n}{2^n} \to 0
\]

so by Sect. Th., \( \frac{3^n}{n!} \to 0 \) as \( n \to \infty \)