

LAST TIME:

If $a_n \xrightarrow{n \rightarrow \infty} L$, then $\{a_n\}$ converges (to L).

If $\lim_{n \rightarrow \infty} a_n$ does not exist, then $\{a_n\}$ diverges.

The Squeeze Theorem

If $a_n < b_n < c_n$ for $n > M$ & $a_n \xrightarrow{n \rightarrow \infty} L$ & $c_n \xrightarrow{n \rightarrow \infty} L$, then $b_n \xrightarrow{n \rightarrow \infty} L$.

A second application of the Squeeze Theorem.

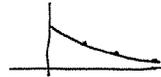
If $|a_n| \xrightarrow{n \rightarrow \infty} 0$ then $a_n \xrightarrow{n \rightarrow \infty} 0$

$$\text{Since } -|a_n| \leq a_n \leq |a_n|$$

An important type of sequence, Geometric Sequence, $\{cr^n\}$, $c \neq 0$

$$\lim_{n \rightarrow \infty} cr^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ c & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1 \end{cases}$$

Proof: if $0 < r < 1$, $r^n \xrightarrow{n \rightarrow \infty} 0$ since $r^x \xrightarrow{x \rightarrow \infty} 0$



if $-1 < r < 0$, $-|r|^n < r^n < |r|^n$ so $r^n \xrightarrow{n \rightarrow \infty} 0$ by the squeeze theorem

if $r = 1$, $cr^n = c$ for all n

if $r = -1$, $cr^n = c(-1)^n$ which oscillates

if $r > 1$, $cr^n \rightarrow \infty$

if $r < -1$, cr^n oscillates & grows unbounded.

Theorem: If f is continuous & $a_n \xrightarrow{n \rightarrow \infty} L$

$$\text{then } \lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right) = f(L)$$

$$\text{Ex) } \frac{n}{n+1} \xrightarrow{n \rightarrow \infty} 1 \quad \text{so} \quad \ln\left(\frac{n}{n+1}\right) \xrightarrow{n \rightarrow \infty} 0$$

The big gun

16 OCT 2015

2//

Definition: $\{a_n\}$ is bounded above if there exists M s.th. $a_n < M$ for all n
 $\{a_n\}$ is bounded below if there exists N s.th. $a_n > N$ for all n
 $\{a_n\}$ is bounded if it is bounded above & below.

Note: convergent sequences are bounded.

Definition: $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all n
 $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all n
 $\{a_n\}$ is monotone if it is increasing or decreasing.

Theorem: Monotone Convergence Theorem.

If $\{a_n\}$ is bounded & monotone, then it converges.

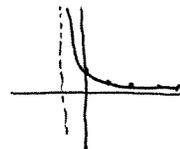
Toy example:

Show $a_n = \frac{1}{n+1}$ is convergent.

a_n is decreasing since $a_n = \frac{1}{n+1} > \frac{1}{n+2} = a_{n+1}$

-OR- could show $f(x) = \frac{1}{x+1}$ is decreasing

$$f'(x) = \frac{-1}{(x+1)^2} < 0 \text{ for all } x$$



a_n is bounded since $0 < \frac{1}{n+1} \leq 1$

so by the M.C.T., $\{a_n\} = \left\{ \frac{1}{n+1} \right\}$ converges.

Material not in text,

Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ so $\binom{3}{2} = \frac{3!}{2!1!} = 3 = \binom{3}{1}$ $\binom{3}{0} = \frac{3!}{3!0!} = \binom{3}{3}$ Recall $0! = 1$

Example $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

A special form

$$(1+x)^n = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \dots + \binom{n}{n} x^n$$

$$= 1 + nx + \frac{n(n-1)}{2} x^2 + \dots + x^n$$

Theorem

A) If $p > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$

B) If $p > 0$, $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$

C) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ \longleftarrow Note: This implies $\sqrt[n]{n^k} \rightarrow 1$ for $k \in \mathbb{N}$,

D) If $|r| < 1$, $\lim_{n \rightarrow \infty} r^n = 0$

E) $\lim_{n \rightarrow \infty} \left(1 + \frac{c}{n}\right)^n = e^c$

which further implies if $p(n)$ is a polynomial in n , then

$$\sqrt[n]{|p(n)|} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof

A) Take $n > \left(\frac{1}{\varepsilon}\right)^{1/p}$ so $\left|\frac{1}{n^p} - 0\right| < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$ as required

B) If $p > 1$, let $z_n = \sqrt[n]{p} - 1$, $z_n > 0$ by binomial theorem

$$1 + n z_n \leq (1 + z_n)^n = p \Rightarrow n z_n \leq p - 1 \Rightarrow 0 \leq z_n \leq \frac{p-1}{n}$$

Since $\frac{p-1}{n} \xrightarrow[n \rightarrow \infty]{} 0$, by the squeeze theorem

$$z_n \xrightarrow[n \rightarrow \infty]{} 0 \Rightarrow \sqrt[n]{p} \xrightarrow[n \rightarrow \infty]{} 1$$

If $p = 1$, obvious

If $p < 1$, let $b_n = \frac{1}{\sqrt[n]{p}} - 1$ the argument is similar to above.

C) Let $z_n = \sqrt[n]{n} - 1$, then $z_n > 0$ by the binomial theorem

$$n = (1 + z_n)^n \geq \frac{n(n-1)}{2} z_n^2 \Rightarrow \frac{2}{n-1} \geq z_n^2 > 0$$

$$\Rightarrow 0 \leq z_n \leq \sqrt{\frac{2}{n-1}}$$

Since $\sqrt{\frac{2}{n-1}} \xrightarrow[n \rightarrow \infty]{} 0$, by squeeze, $z_n \xrightarrow[n \rightarrow \infty]{} 0$, i.e. $\sqrt[n]{n} \xrightarrow[n \rightarrow \infty]{} 1$

— OR —

$$\lim_{n \rightarrow \infty} n^{1/n} = \lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} e^{\frac{1}{x} \ln x} = e^{\lim_{x \rightarrow \infty} \frac{\ln x}{x}} \stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{1}{x}} = e^0 = 1$$

D) See earlier proof

$$\begin{aligned} E) \lim_{n \rightarrow \infty} \left(1 + \frac{c}{n}\right)^n &= \lim_{x \rightarrow \infty} \left(1 + \frac{c}{x}\right)^x = e^{\lim_{x \rightarrow \infty} x \ln\left(1 + \frac{c}{x}\right)} = e^{\lim_{x \rightarrow \infty} \left[\frac{\ln\left(1 + \frac{c}{x}\right)}{\frac{1}{x}}\right]} \\ &= e^{\lim_{x \rightarrow \infty} \left(\frac{\frac{1}{1 + \frac{c}{x}} \cdot \frac{-c}{x^2}}{-\frac{1}{x^2}}\right)} = e^{\lim_{x \rightarrow \infty} \frac{c}{1 + \frac{c}{x}}} = e^c \end{aligned}$$

□