

Sequences Review

If $z_n \rightarrow c$ then $\{z_n\}$ converges for any c .

Geometric Sequences

if $|r| < 1$, $r^n \rightarrow 0$ as $n \rightarrow \infty$

Bounded

All convergent sequences are bounded, but not all bounded are convergent.

A good example is $\{1, 0, 1, 0, 1, 0, \dots\}$

Increasing, Decreasing, Monoton

A monotone sequence need not be convergent, e.g. $\{1, 2, 3, 4, \dots\}$

or $\{5, 4, 3, 2, 1, 0, -1, \dots\}$

But, A bounded monotone sequence converges!

Finding Limits

Standard calc is for rational expressions

$$\frac{n^2 + 3n}{\sqrt{n^4 + 2n^2 + 1}} \leftarrow \text{divide by highest power in the denominator,}$$

in this case $n^2 = \sqrt{n^4}$ giving

$$\frac{1 + \frac{3}{n}}{\sqrt{1 + \frac{2}{n^2} + \frac{1}{n^4}}} \longrightarrow 1 \text{ as } n \rightarrow \infty$$

so $\ln \left(\frac{n^2 + 3n}{\sqrt{n^4 + 2n^2 + 1}} \right) \xrightarrow{n \rightarrow \infty} \ln(1) = 0$

The series $\sum_{n=1}^{\infty} z_n$ converges (to S) if the sequence of partial sums

$$\{S_N\} = \left\{ \sum_{n=1}^N z_n \right\} \text{ converges (to } S \text{)}.$$

Last Time we talked about two types of series we could sum (i.e. find the limit)

1) Geometric

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r} \quad \text{OR} \quad \sum_{n=?}^{\infty} cr^n = \frac{\text{first term}}{1-r} \quad \text{for } |r| < 1$$

$$\text{Ex]} \quad \sum_{n=0}^{\infty} \frac{3(-1)^n}{2^{2^n}} = \sum_{n=0}^{\infty} 3 \left(\frac{-1}{4}\right)^n = \frac{3}{1 + \frac{1}{4}} = \frac{3}{5/4} = \frac{12}{5}$$

2) Telescoping

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \frac{2}{\cancel{n(n+2)}} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) \quad \text{so}$$

$$S_1 = 1 - \frac{1}{3}$$

$$S_N \rightarrow \frac{3}{2}$$

$$S_2 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$$

$$\text{so} \quad \sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{3}{2}$$

$$S_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$$

:

$$S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

Since finite sums & limits are linear, so are series, i.e.

$$\sum c z_n = c \sum z_n \quad \& \quad \sum (z_n + b_n) = \underbrace{\sum z_n + \sum b_n}_{\text{provided these converge.}}$$

Linearity allows us to split into convergent pieces, so

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$$\sum_{n=0}^{\infty} \frac{2+3^n}{4^n} = \sum_{n=0}^{\infty} 2 \left(\frac{1}{4}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{2}{1-\frac{1}{4}} + \frac{1}{1-\frac{3}{4}} = 2 \cdot \frac{4}{3} + 4$$

But

$$0 = \sum 0 = \sum (1-1) \neq \underbrace{\sum 1 - \sum 1}_{\text{these both diverge!}}$$

Although finding the sum of a series is often hard, we want to start building some tools to talk about series.

Theorem: If $\sum a_k$ converges, $a_k \xrightarrow[k \rightarrow \infty]{} 0$.

Proof: $a_k = S_k - S_{k-1}$, $\sum a_k$ converges so $S_k \rightarrow S$ as $k \rightarrow \infty$
so $S_{k-1} \rightarrow S$ as $k \rightarrow \infty$

taking limits gives $a_k \xrightarrow[k \rightarrow \infty]{} 0$.

We typically use this result in the form of The Test for Divergence:

If $a_k \not\rightarrow 0$ as $k \rightarrow \infty$ then $\sum a_k$ diverges

Note: we are saying the terms going to zero is a necessary condition,
it is however, not sufficient.

Theorem: The Harmonic Series diverges,

$$\text{i.e. } \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

Proof:

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right)$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right)$$

⋮

$$S_{2^n} > 1 + \frac{n}{2} \quad \text{Since } 1 + \frac{n}{2} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

$$S_{2^n} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

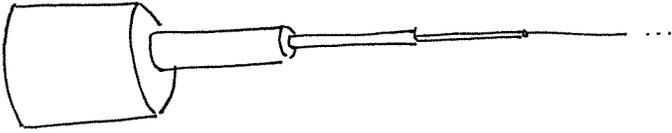
∴ the series diverges!

Note: $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

So, never ever ever ever write on your paper

since $a_k \longrightarrow 0$ $\sum a_k$ converges,

it is nonsense of the worst type!



Consider the infinite wedding cake with each layer 1' in length & $r = 1, \frac{1}{2}, \frac{1}{3}, \dots$

Then the volume is

$$V = \sum_{n=1}^{\infty} \pi (1) \left(\frac{1}{n}\right)^2$$

$$= \pi \left[1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \frac{1}{5 \cdot 5} + \dots \right]$$

This type of comparison is covered in §10.3

$$< \pi \left[1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \frac{1}{5 \cdot 4} + \dots \right] = \pi \left[1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \right] = \pi (1+1) = 2\pi$$

(Note: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64\dots$)

So finite volume.

What about surface area?

• Annular regions



$$\pi (1^2 - (\frac{1}{2})^2) + \pi ((\frac{1}{2})^2 - (\frac{1}{3})^2) + \pi ((\frac{1}{3})^2 - (\frac{1}{4})^2) + \dots = \sum_{n=1}^{\infty} \pi \left(\left(\frac{1}{n}\right)^2 - \left(\frac{1}{n+1}\right)^2 \right)$$

which again telescopes

$$S_1 = \pi \left(1 - \frac{1}{4}\right)$$

$$S_2 = \pi \left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) \right] = \pi \left(1 - \frac{1}{9}\right)$$

⋮

$$S_n = \pi \left(1 - \frac{1}{(n+1)^2}\right) \longrightarrow \pi (1) \text{ which makes sense}$$



"Side" surface area

$$2\pi(1)(1) + 2\pi(1)\left(\frac{1}{2}\right) + 2\pi(1)\left(\frac{1}{3}\right) + \dots = \sum_{n=1}^{\infty} 2\pi \frac{1}{n} = 2\pi \sum_{n=1}^{\infty} \frac{1}{n}$$

We consider the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ * The Harmonic Series *

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right)$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right)$$

⋮

$$S_{2^N} > 1 + N\left(\frac{1}{2}\right)$$

$$\text{Since } 1 + \frac{N}{2} \longrightarrow \infty \text{ as } N \rightarrow \infty$$

$$S_{2^N} \longrightarrow \infty \text{ as } N \rightarrow \infty$$

∴ the series diverges!

i.e. there is infinite surface area.