7.1 Integration by Parts

We know from the Product Rule that

$$u(x)\,v'(x) + u'(x)\,v(x) = \frac{d}{dx}\left(u(x)\,v(x)\right).$$

Integrating with respect to x gives

$$\int u(x)\,v'(x)\,dx + \int v(x)\,u'(x)\,dx = u(x)v(x).$$

Moving the second integral to the other side of the equation & using the notation

$u = u(x)$ so $du = u'(x)\,dx$ and $v = v(x)$ so $dv = v'(x)\,dx$ gives

$$\int u\,dv = uv - \int v\,du$$

Important classes of examples.

1) Reduction of order (polynomials)

A) $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \sin x$

B) $\int (x^2+2)\,e^{3x}\,dx = \frac{1}{3}(x^2+2)e^{3x} - \int \frac{2}{3}x e^{3x}\,dx$

$u = x^2+2$ $\quad dv = e^{3x}\,dx$

$du = 2x\,dx$ $\quad v = \frac{1}{3}e^{3x}$

$u = \frac{2}{3}x$ $\quad dv = e^{3x}\,dx$

$du = \frac{2}{3}dx$ $\quad v = \frac{1}{3}e^{3x}$

$$= \frac{1}{3}(x^2+2)e^{3x} - \left(\frac{2}{9}x e^{3x} - \int \frac{2}{9}e^{3x}\,dx\right)$$

$$= \frac{1}{3}(x^2+2)e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C$$

C) $\int (x^3+3x)\sin x\,dx$ requires <u>three</u> applications of I.B.P.

etc.

2) Logs & other inverse functions ("integrating by differentiation")

A) $\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

B) $\int \frac{\ln x}{x} dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

$u = \ln x$

$du = \frac{1}{x} dx$

* Don't do IBP when a u-sub works

C) $\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \frac{1}{2}\int u^{-1/2} du$

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$u = 1 - x^2$

$du = -2x \, dx$

$= x \arcsin x + \sqrt{u} + C$

$= x \arcsin x + \sqrt{1-x^2} + C$

A puzzle

$\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx$ so $0 \stackrel{?}{=} 1$

$u = \frac{1}{x} \qquad dv = dx$

$du = -\frac{1}{x^2} dx \qquad v = x$

3) Definite Integrals

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

A) $\int_0^\pi x \sin x \, dx = -x \cos x \Big|_0^\pi + \int_0^\pi \cos x \, dx = -\pi(-1) + \sin x \Big|_0^\pi = \pi$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

B) $\int_0^1 y \arctan y \, dy = \frac{y^2}{2} \arctan y \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{y^2}{1+y^2} dy = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\int_0^1 \left(\frac{y^2+1}{y^2+1} - \frac{1}{y^2+1}\right) dy$

$= \frac{\pi}{8} - \frac{1}{2}\left(y - \arctan y\right)\Big|_0^1$

$= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$

$= \frac{\pi}{4} - \frac{1}{2}$

$u = \arctan y \qquad dv = y \, dy$

$du = \frac{1}{1+y^2} dy \qquad v = \frac{1}{2}y^2$

4) "Tricky" Integrals

A) $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$

$u = \sin x \quad dv = e^x dx \qquad u = \cos x \quad dv = e^x dx$

$du = \cos x \, dx \quad v = e^x \qquad du = -\sin x \, dx \quad v = e^x$

so $\quad 2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$

$$\int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

B) $\int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx = -\sin x \cos x + \int dx - \int \sin^2 x \, dx$

($\cos^2 x = 1 - \sin^2 x$)

$u = \sin x \quad dv = \sin x \, dx$

$du = \cos x \, dx \quad v = -\cos x$

so $\quad 2 \int \sin^2 x \, dx = x - \sin x \cos x + C$

$$\int \sin^2 x \, dx = \frac{1}{2} (x - \sin x \cos x) + C$$

This type of trick is used to derive many of the trigonometric reduction identities, for example:

5) $\int \sec^3 x \, dx$, Reduction of order of trig integrals

This integral will come up often. For example, the arc length of a parabola.

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$

($\tan^2 x = \sec^2 x - 1$)

$u = \sec x \quad dv = \sec^2 x \, dx$

$du = \sec x \tan x \quad v = \tan x$

so $\quad 2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C$

$$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$