

# Math 172 - Trigonometric Integrals

## Section 7.2 Done Right

### 1. INTEGRALS OF THE FORM $\int \sin^m \theta \cos^n \theta d\theta$ .

- If the power of cosine is odd and positive, save one cosine factor, use  $\cos^2 \theta = 1 - \sin^2 \theta$  to express the remaining factors in terms of sine, and substitute  $u = \sin \theta$  so that  $du = \cos \theta d\theta$ . For example

$$\begin{aligned}
 \int \cos^3 x dx &= \int \cos^2 x \cos x dx = \int (1 - \sin^2 x) \cos x dx \\
 &\stackrel{u=\sin x}{=} \int_{du=\cos x dx} (1 - u^2) du \\
 &= u - \frac{1}{3}u^3 + C \\
 &= \sin x - \frac{1}{3} \sin^3 x + C.
 \end{aligned}$$

- If the power of sine is odd and positive, save one sine factor, use  $\sin^2 \theta = 1 - \cos^2 \theta$  to express the remaining factors in terms of cosine, and substitute  $u = \cos \theta$  so that  $du = -\sin \theta d\theta$ . For example

$$\begin{aligned}
 \int \sin^5 2x \cos^2 2x dx &= \int (\sin^2 2x)^2 \cos^2 2x \sin 2x dx \\
 &= \int (1 - \cos^2 2x)^2 \cos^2 2x \sin 2x dx \\
 &\stackrel{u=\cos 2x}{=} \int_{du=-2 \sin 2x dx} (1 - u^2)^2 u^2 du \\
 &= -\frac{1}{2} \int (u^2 - 2u^4 + u^6) du \\
 &= -\frac{1}{6}u^3 + \frac{1}{5}u^5 - \frac{1}{14}u^7 + C \\
 &= -\frac{1}{6} \cos^3 2x + \frac{1}{5} \cos^5 2x - \frac{1}{14} \cos^7 2x + C.
 \end{aligned}$$

- If both are odd and positive, use either of the above.
- If both are even and positive, use the half-angle identities

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \text{or} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

to rewrite until you have odd powers. For example

$$\begin{aligned}
 \int \sin^4 3x dx &= \int (\sin^2 3x)^2 dx = \int \left( \frac{1 - \cos 6x}{2} \right)^2 dx \\
 &= \frac{1}{4} \int (1 - 2 \cos 6x + \cos^2 6x) dx \\
 &= \frac{1}{4} \int \left( 1 - 2 \cos 6x + \frac{1 + \cos 12x}{2} \right) dx \\
 &= \frac{1}{8} \int (3 - 4 \cos 6x + \cos 12x) dx \\
 &= \frac{3}{8}x - \frac{1}{12} \sin 6x + \frac{1}{96} \sin 12x + C.
 \end{aligned}$$

2. INTEGRALS OF THE FORM  $\int \tan^m \theta \sec^n \theta d\theta$ .

- We have discussed the three basic integrals.

$$\begin{aligned} \circ \int \tan \theta d\theta &= \ln |\sec \theta| + C \\ \circ \int \sec^3 \theta d\theta &= \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C \end{aligned}$$

- If the power of secant is even and positive ( $n = 2, 4, 6, \dots$ ), save a factor of  $\sec^2 \theta$ , use  $\sec^2 \theta = 1 + \tan^2 \theta$  to express the remaining factors in terms of tangent, and substitute  $u = \tan \theta$  so that  $du = \sec^2 \theta d\theta$ . For example

$$\begin{aligned} \int 4 \tan^4 4x \sec^4 4x dx &= \int 4 \tan^4 4x \sec^2 4x \sec^2 4x dx \\ &= \int \tan^4 4x (1 + \tan^2 4x) 4 \sec^2 4x dx \\ &\stackrel{u=\tan 4x}{=} \int u^4 (1 + u^2) du \\ &= \int (u^4 + u^6) du \\ &= \frac{1}{5} u^5 + \frac{1}{7} u^7 + C \\ &= \frac{1}{5} \tan^5 4x + \frac{1}{7} \tan^7 4x + C. \end{aligned}$$

- If there is at least one secant and the power of tangent is odd and positive, save a factor of  $\sec \theta \tan \theta$ , use  $\tan^2 \theta = \sec^2 \theta - 1$  to express the remaining factors in terms of secant, and substitute  $u = \sec \theta$  so that  $du = \sec \theta \tan \theta d\theta$ . For example

$$\begin{aligned} \int \tan^5 5x \sec 5x dx &= \int (\tan^2 5x)^2 \sec 5x \tan 5x dx \\ &= \int (\sec^2 5x - 1)^2 \sec 5x \tan 5x dx \\ &\stackrel{u=\sec 5x}{=} \int (u^2 - 1)^2 du \\ &= \frac{1}{5} \int (u^4 - 2u^2 + 1) du \\ &= \frac{1}{25} u^5 - \frac{2}{15} u^3 + \frac{1}{5} u + C \\ &= \frac{1}{25} \sec^5 5x - \frac{2}{15} \sec^3 5x + \frac{1}{5} \sec 5x + C. \end{aligned}$$

- If neither of the above apply, you have an adventure on your hands. You can try to use the identity  $\tan^2 \theta + 1 = \sec^2 \theta$  to rewrite it. You may have to convert everything to  $\sec \theta$  and use integration by parts in a manner similar to the way we derived the integral of  $\sec^3 \theta$ . Two random examples follow.

$$\begin{aligned} \int 6 \tan^2 6x \sec 6x dx &\stackrel{u=6x}{=} \int (\sec^2 u - 1) \sec u du = \int (\sec^3 u - \sec u) du \\ &= \left[ \frac{1}{2} (\sec u \tan u + \ln |\sec u + \tan u|) - \ln |\sec u + \tan u| \right] + C \\ &= \frac{1}{2} (\sec u \tan u - \ln |\sec u + \tan u|) + C \\ &= \frac{1}{2} (\sec 6x \tan 6x - \ln |\sec 6x + \tan 6x|) + C \end{aligned}$$

$$\begin{aligned}
\int \tan^4 7x \, dx &= \int (\sec^2 7x - 1)^2 \, dx \\
&= \int (\sec^4 7x - 2 \sec^2 7x + 1) \, dx \\
&= \int (\sec^2 7x - 2) \sec^2 7x \, dx + \int 1 \, dx \\
&= \int (\tan^2 7x - 1) \sec^2 7x \, dx + x \\
&\stackrel{u=\tan 7x}{=} \frac{1}{7} \int (u^2 - 1) \, du + x \\
&= \frac{1}{21} u^3 - \frac{1}{7} u + x + C \\
&= \frac{1}{21} \tan^3 7x - \frac{1}{7} \tan 7x + x + C
\end{aligned}$$

3. INTEGRALS OF THE FORM  $\int \cot^m \theta \csc^n \theta \, d\theta$  are treated in a manner similar to the previous case.

4. INTEGRALS WITH ARGUMENTS THAT DO NOT MATCH. In the previous examples, it was critical that the arguments of the functions matched. Things are trickier when the arguments differ.

- For integrals involving  $\sin 2\theta$  or  $\cos 2\theta$  we can make use of the double-angle identities below.
  - $\sin 2\theta = 2 \sin \theta \cos \theta$
  - $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$

Return to the second example. Just because you see a  $2\theta$  does not mean you should use the double-angle identities; using the identities in that example would have greatly increased the work required. However, you should use them when the arguments do not match. For example

$$\int \sin 2x \cos x \, dx = \int 2 \sin x \cos^2 x \, dx \stackrel{u=\cos x}{=} -2 \int u^2 \, du = -\frac{2}{3} u^3 + C = -\frac{2}{3} \cos^3 x + C$$

and

$$\int \cos 2x \tan x \, dx = \int (2 \cos^2 x - 1) \frac{\sin x}{\cos x} \, dx = \int (2 \cos x \sin x - \tan x) \, dx = \sin^2 x - \ln |\sec x| + C.$$

- For integrals involving  $\sin nx \cos mx$ ,  $\cos nx \cos mx$ , or  $\sin nx \sin mx$ , use the product to sum identities.

- $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$
- $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
- $\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

For example

$$\begin{aligned}
\int \sin 5x \cos 2x \, dx &= \frac{1}{2} \int (\sin(5x + 2x) + \sin(5x - 2x)) \, dx \\
&= \frac{1}{2} \int (\sin 7x + \sin 3x) \, dx \\
&= -\frac{1}{14} \cos 7x - \frac{1}{6} \cos 3x + C.
\end{aligned}$$

- For other integrals with mismatched arguments, find a computer, it will be a better use of your time.

## Math 172 - Trigonometric Integrals - Supplemental Examples

5. INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS WITH A NONPOSITIVE OR NONINTEGER POWER. The same basic rules apply. Two examples follow;

$$\int \frac{\cos x}{\sqrt{\sin x}} dx \stackrel{u=\sin x}{du=\cos x dx} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{\sin x} + C$$

and

$$\begin{aligned} \int \tan x \sin^4 x dx &= \int \frac{\sin x}{\cos x} \sin^4 x dx = \int \frac{\sin^5 x}{\cos x} dx = \int \frac{(\sin^2 x)^2 \sin x}{\cos x} dx \\ &= \int \frac{(1 - \cos^2 x)^2}{\cos x} \sin x dx \\ &\stackrel{u=\cos x}{du=-\sin x dx} - \int \frac{(1 - u^2)^2}{u} du \\ &= - \int \frac{1 - 2u^2 + u^4}{u} du = - \int \left( \frac{1}{u} - 2u + u^3 \right) du \\ &= - \ln |u| + u^2 - \frac{1}{4}u^4 + C \\ &= \ln |\sec x| + \cos^2 x - \frac{1}{4} \cos^4 x + C. \end{aligned}$$

6.  $\int \sec^3 \theta d\theta$ . Surprisingly, this is a remarkably important integral. We use integration by parts with

$$\begin{array}{ll} u = \sec \theta & dv = \sec^2 \theta d\theta \\ du = \sec \theta \tan \theta d\theta & v = \tan \theta \end{array}$$

after the second equal sign.

$$\begin{aligned} \int \sec^3 \theta d\theta &= \int \sec \theta \sec^2 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta \\ &= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta \\ &= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta \\ &= \sec \theta \tan \theta + \int \sec \theta d\theta - \int \sec^3 \theta d\theta \\ &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| - \int \sec^3 \theta d\theta \end{aligned}$$

Combining the integrals gives

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| + C$$

and hence

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C$$

In the example above we related  $\int \sec^3 \theta d\theta$  to  $\int \sec \theta d\theta$ . A similar strategy works for relating  $\int \sec^5 \theta d\theta$  to  $\int \sec^3 \theta d\theta$ . In general, the same strategy works to relate  $\int \sec^{2n+1} \theta d\theta$  to  $\int \sec^{2n-1} \theta d\theta$ .