

MATH 172 Exam 1 Review Solutions

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$$1. A. \int_0^3 t^2 \sqrt{1+t} dt = \int_1^4 (u-1)^2 \sqrt{u} du = \int_1^4 (u^{5/2} - 2u^{3/2} + u^{1/2}) du = \left. \frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right|_1^4$$

let $u = 1+t$ $\frac{x}{0} \mapsto \frac{u}{1}$
 $du = dt$ $\frac{3}{3} \quad \quad \quad \frac{4}{4}$

$$= \frac{2}{7} (128-1) - \frac{4}{5} (32-1) + \frac{2}{3} (8-1) \leftarrow \text{stop here}$$

$$= \frac{1696}{105}$$

$$B. \int \frac{dy}{y^2+4} = \frac{1}{4} \int \frac{dy}{(\frac{y}{2})^2+1} = \frac{1}{2} \int \frac{du}{u^2+1} = \frac{1}{2} \arctan\left(\frac{y}{2}\right) + C$$

let $u = \frac{y}{2}$ so $du = \frac{1}{2} dy$

$$C. \int_0^{\pi/2} x \cos x dx = x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x dx = \frac{\pi}{2} + \cos x \Big|_0^{\pi/2} = \frac{\pi}{2} - 1$$

$u = x$ $dv = \cos x dx$
 $du = dx$ $v = \sin x$

$$D. \int x \arctan x dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{x^2+1} dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} \right) dx$$

$x^2 = x^2 + 1 - 1$ $= 1$

$u = \arctan x$ $dv = x dx$

$du = \frac{1}{1+x^2} dx$ $v = \frac{x^2}{2}$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C$$

$$2. A. \int_1^{e^\pi} \frac{\sin(\ln t)}{t} dt = \int_0^\pi \sin u du = -\cos u \Big|_0^\pi = 2$$

let $u = \ln t$ $\frac{t}{1} \mapsto \frac{u}{0}$
 $du = \frac{1}{t} dt$ $e^\pi \quad \quad \quad \pi$

$$B. \int \frac{dy}{\sqrt{2-y^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1-(\frac{y}{\sqrt{2}})^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin\left(\frac{y}{\sqrt{2}}\right) + C$$

let $u = \frac{y}{\sqrt{2}}$ so $du = \frac{1}{\sqrt{2}} dy$

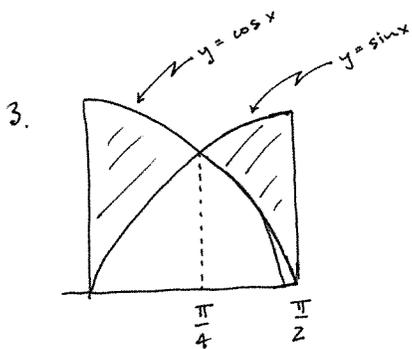
2.c. $\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

$u = \ln x \quad dv = x \, dx$

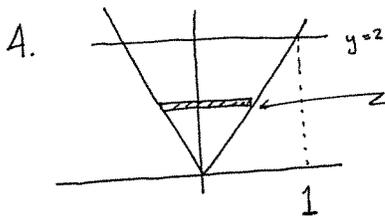
$du = \frac{1}{x} \, dx \quad v = \frac{x^2}{2}$

D. $\int_0^3 \frac{1+2x}{9+x^2} \, dx = \frac{1}{9} \int_0^3 \frac{dx}{1+(\frac{x}{3})^2} + \int_0^3 \frac{2x}{9+x^2} = \frac{1}{3} \arctan\left(\frac{x}{3}\right) \Big|_0^3 + \ln(9+x^2) \Big|_0^3 = \frac{\pi}{12} + \ln 2$

$\ln 18 - \ln 9 = \ln 2$
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3. Area = $\int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx$
 $= 2 \int_0^{\pi/4} (\cos x - \sin x) \, dx$ by symmetry
 $= 2 (\sin x + \cos x) \Big|_0^{\pi/4} = 2(\sqrt{2}-1)$

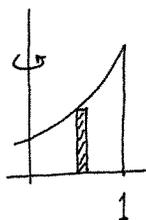


4. Volume of slice = (side)² Δy
 $= (2x)^2 \Delta y$
 $= y^2 \Delta y$

so $V = \int_0^2 y^2 \, dy = \frac{8}{3}$

5. Average Value = $\frac{1}{\pi/4 - 0} \int_0^{\pi/4} \tan x \, dx = \frac{4}{\pi} \ln |\sec x| \Big|_0^{\pi/4} = \frac{4}{\pi} \ln \sqrt{2}$

6. Using washers would require two integrals & would involve $1 - (\ln y)^2$, which is ugly. We should use shells.



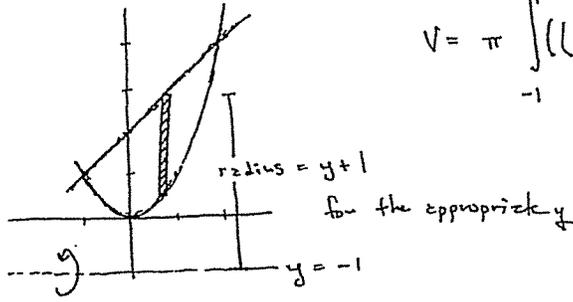
$2\pi \int_0^1 x e^x \, dx = 2\pi \left(x e^x \Big|_0^1 - \int_0^1 e^x \, dx \right) = 2\pi (e - e^x \Big|_0^1) = 2\pi$

$u = x \quad dv = e^x \, dx$
 $du = dx \quad v = e^x$

7. $y = x^2$, $y = x + 2$

Shells would require two integrals, let's use disks.

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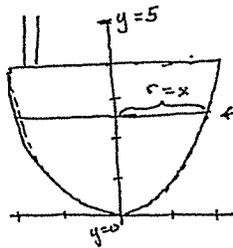
$$V = \pi \int_{-1}^2 ((x+2+1)^2 - (x^2+1)^2) dx = \pi \int_{-1}^2 (8 + 6x - x^2 - x^4) dx$$

$$= \pi \left(8x + 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_{-1}^2$$

$$= \pi \left((16 + 12 - \frac{8}{3} - \frac{32}{5}) - (-8 + 3 + \frac{1}{3} + \frac{1}{5}) \right) \leftarrow \text{stop}$$

$$= \frac{117}{5} \pi$$

8.



Work for this slice

$$= \rho g \pi r^2 (y + 5) \Delta y = \rho g \pi x^2 (5 - y) \Delta y$$

$$= \rho g \pi (5y - y^2) \Delta y$$

$$\text{Work} = \rho g \pi \int_0^4 (5y - y^2) dy = \rho g \pi \left(\frac{5}{2} y^2 - \frac{y^3}{3} \right) \Big|_0^4 = \rho g \pi \left(40 - \frac{64}{3} \right) = \frac{56 \pi \rho g}{3}$$

9. A. $\int \frac{t dt}{\sqrt{4-t^2}} = -\frac{1}{2} \int u^{-1/2} du = -u^{1/2} + C = C - \sqrt{4-t^2}$

let $u = 4-t^2$
 $du = -2t dt$

B. $\int_{1/8}^{1/4} \frac{dy}{\sqrt{1-16y^2}} = \int_{1/2}^1 \frac{du}{\sqrt{1-u^2}} = \frac{1}{4} \arcsin u \Big|_{1/2}^1 = \frac{1}{4} \left(\frac{\pi}{2} - \frac{\pi}{6} \right) = \frac{1}{4} \cdot \frac{\pi}{3} = \frac{\pi}{12}$

let $u = 4y$ $\frac{y}{1/8} \rightarrow \frac{u}{1/2}$
 $du = 4dy$ $\frac{1}{4}$ 1

C. $\int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} (\ln x)^2 + C$

$u = \ln x$
 $du = \frac{1}{x} dx$

Note: we can also use parts

$\int \frac{\ln x}{x} dx = (\ln x)^2 - \int \frac{\ln x}{x} dx$ so $2 \int \frac{\ln x}{x} dx = (\ln x)^2 + C$

$u = \ln x$ $dv = \frac{1}{x} dx$
 $du = \frac{1}{x} dx$ $v = \ln x$

so $\int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + C$

D. $\int_0^{\pi/4} x \sec^2 x dx = x \tan x \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x dx = \frac{\pi}{4} - \ln |\sec x| \Big|_0^{\pi/4} = \frac{\pi}{4} - \ln \sqrt{2}$

$u = x$ $dv = \sec^2 x dx$
 $du = dx$ $v = \tan x$

10. A. $\int t \sqrt{2-t} dt = - \int (2-u) \sqrt{u} du = \int (u^{3/2} - 2u^{1/2}) du = \frac{2}{5} (2-t)^{5/2} - \frac{4}{3} (2-t)^{3/2} + C$

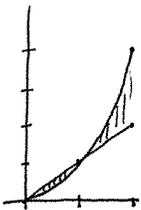
let $u = 2-t$
 $du = -dt$

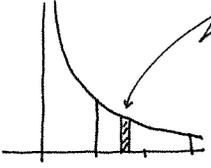
10. B $\int_0^1 \frac{dy}{3y^2+1} = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{du}{u^2+1} = \frac{1}{\sqrt{3}} \arctan u \Big|_0^{\sqrt{3}} = \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - 0 \right) = \frac{\pi}{3\sqrt{3}}$

let $u = \sqrt{3}y$ $\frac{y}{0} \mapsto \frac{u}{0}$
 $du = \sqrt{3} dy$ $1 \quad \sqrt{3}$

C. $\int_0^1 \frac{1+2x}{\sqrt{1-x^2}} dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{2x}{\sqrt{1-x^2}} dx = \arcsin x \Big|_0^1 - \int_1^0 u^{-1/2} du = \frac{\pi}{2} + 2\sqrt{u} \Big|_1^0 = \frac{\pi}{2} + 2$
 $\frac{x}{0} \mapsto \frac{u}{1}$
 $1 \quad 0$
 $u = 1-x^2$
 $du = -2x dx$

D. $\int x^2 \sin 2x dx = -\frac{x^2}{2} \cos 2x + \int x \cos 2x dx = -\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x - \int \frac{1}{2} \sin 2x dx$
 $u = x \quad dv = \sin 2x dx$
 $du = dx \quad v = -\frac{1}{2} \cos 2x$
 $u = x \quad dv = \cos 2x dx$
 $du = dx \quad v = \frac{1}{2} \sin 2x$
 $= -\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C$

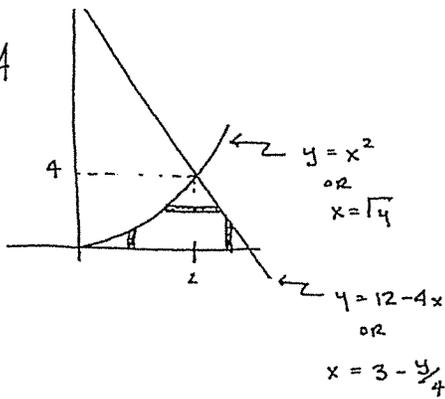
11.  $A = \int_0^1 (x-x^2) dx + \int_1^2 (x^2-x) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2 = \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right) = 1$
 Stop here.

12.  Volume of this slice = $\frac{\pi}{2} (\text{radius})^2 \Delta x$ with radius = $\frac{y}{2} = \frac{1}{2x}$
 $= \frac{\pi}{8} \cdot \frac{1}{x^2} \Delta x$
 Volume = $\frac{\pi}{8} \int_1^3 x^{-2} dx = -\frac{\pi}{8} \cdot \frac{1}{x} \Big|_1^3 = \frac{\pi}{8} \left(1 - \frac{1}{3} \right) = \frac{\pi}{12}$

13. Average = $\frac{1}{\frac{\pi}{n} - 0} \int_0^{\frac{\pi}{n}} \sin(nx) dx = \frac{n}{\pi} \cdot \frac{1}{n} \int_0^{\pi} \sin u du = \frac{1}{\pi} (-\cos u) \Big|_0^{\pi} = \frac{2}{\pi}$

let $u = nx$ $\frac{x}{0} \mapsto \frac{u}{0}$
 $du = n dx$ $\frac{\pi}{n} \quad \pi$

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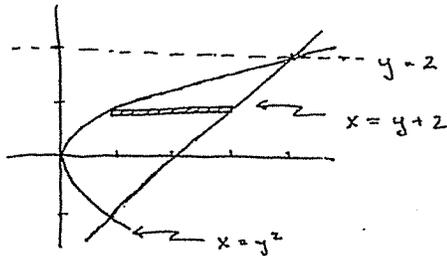


$$V_{\text{Disk}} = \pi \int_0^1 (x^2)^2 dx + \pi \int_1^4 (12 - 4x)^2 dx$$

$$V_{\text{Shell}} = 2\pi \int_0^4 (y) \left(3 - \frac{y}{4} - \sqrt{y} \right) dy$$

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15.



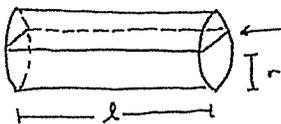
Washers would require two integrals, so we use shells.

$$V = 2\pi \int_{-1}^2 (2 - y) (y + 2 - y^2) dy = 2\pi \int_{-1}^2 (4 - 3y^2 + y^3) dy$$

$$= 2\pi \left(4y - y^3 + \frac{y^4}{4} \right) \Big|_{-1}^2 = 2\pi \left(8 - 8 + 4 - \left(-4 + 1 + \frac{1}{4} \right) \right)$$

$$= 2\pi \left(\frac{27}{4} \right) = \frac{27\pi}{2}$$

16.



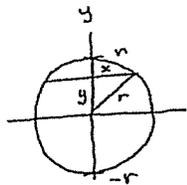
Work for this slice

$$= \rho g (\text{width}) (\text{length}) (\text{distance}) \Delta y$$

$$= \rho g (2x) (l) (r - y) \Delta y$$

$$= \rho g (2\sqrt{r^2 - y^2}) l (r - y) \Delta y$$

$$= 2\rho g l (r - y) \sqrt{r^2 - y^2} \Delta y$$



$$x^2 + y^2 = r^2$$

$$\text{so } x = \sqrt{r^2 - y^2}$$

$$W = 2\rho g l \int_{-r}^r (r - y) \sqrt{r^2 - y^2} dy = 2\rho g l \left[\int_{-r}^r r \sqrt{r^2 - y^2} dy - \int_{-r}^r y \sqrt{r^2 - y^2} dy \right]$$

\leftarrow Area of semi-circle
 \leftarrow odd functions so = 0

$$= 2\rho g l r \int_{-r}^r \sqrt{r^2 - y^2} dy = 2\rho g l r \left(\frac{\pi r^2}{2} \right) = \pi \rho g l r^3$$

Additional Problems

$$1. A. \int \frac{x^2}{x+1} dx = \int \frac{(u-1)^2}{u} du = \int \frac{u^2 - 2u + 1}{u} du = \int \left(u - 2 + \frac{1}{u}\right) du$$

let $u = x+1$ so $u-1 = x$
 $du = dx$

$$= \frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| + C$$

$$B. \int_0^1 \frac{x^2}{1+x^6} dx = \frac{1}{3} \int_0^1 \frac{du}{1+u^2} = \frac{1}{3} \arctan u \Big|_0^1 = \frac{1}{3} \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$$

$u = x^3$ $du = 3x^2 dx$

$\frac{x}{0}$	\rightarrow	$\frac{u}{0}$
$\frac{x}{1}$		$\frac{u}{1}$

$$C. \int \arcsin x dx = x \arcsin x - \int \frac{x dx}{\sqrt{1-x^2}} = x \arcsin x + \frac{1}{2} \int u^{-1/2} du$$

$u = \arcsin x$ $dv = dx$

$du = \frac{dx}{\sqrt{1-x^2}}$ $v = x$

let $u = 1-x^2$
 $du = -2x dx$

$= x \arcsin x + \sqrt{1-x^2} + C$

$$D. \int \frac{\ln \sqrt{x}}{\sqrt{x}} dx = 2 \int \ln t dt = 2 [t \ln t - \int dt] = 2 [t \ln t - t] + C$$

let $t = \sqrt{x}$
 $dt = \frac{1}{2\sqrt{x}} dx$

$u = \ln t$ $dv = dt$
 $du = \frac{1}{t} dt$ $v = t$

$= 2\sqrt{x} (\ln \sqrt{x} + 1) + C$

- or -

$= \sqrt{x} (\ln x - 2) + C$

$$2. A. \int \frac{x^7 + x^3}{4+x^8} dx = \int \frac{x^7}{4+x^8} dx + \frac{1}{4} \int \frac{x^3}{1+(\frac{x^4}{2})^2} dx = \frac{1}{8} \ln(4+x^8) + \frac{1}{8} \arctan\left(\frac{x^4}{2}\right) + C$$

$u = 4+x^8$

$du = 8x^7 dx$

$u = \frac{x^4}{2}$

$du = 2x^3 dx$

$$2. B. \int e^{2x} \sin 3x \, dx = \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx$$

$$\begin{array}{ll} u = \sin 3x & dv = e^{2x} \, dx \\ du = 3 \cos 3x \, dx & v = \frac{1}{2} e^{2x} \end{array} \quad \left. \vphantom{\begin{array}{ll} u = \sin 3x & dv = e^{2x} \, dx \\ du = 3 \cos 3x \, dx & v = \frac{1}{2} e^{2x} \end{array}} \right\} \begin{array}{ll} u = \cos 3x & dv = e^{2x} \, dx \\ du = -3 \sin 3x \, dx & v = \frac{1}{2} e^{2x} \end{array}$$

$$= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \left[\frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx \right]$$

$$= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} \int e^{2x} \sin 3x \, dx$$

$$\text{so } \frac{13}{4} \int e^{2x} \sin 3x \, dx = \frac{1}{4} (e^{2x} \sin 3x - 3 e^{2x} \cos 3x) + C$$

$$\int e^{2x} \sin 3x \, dx = \frac{1}{13} e^{2x} (\sin 3x - 3 \cos 3x) + C$$

$$2. c. \int x^2 \arcsin x \, dx = \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} \, dx = \frac{x^3}{3} \arcsin x + \frac{1}{6} \int \frac{(1-u) \, du}{\sqrt{u}}$$

$$\begin{array}{ll} u = \arcsin x & dv = x^2 \, dx \\ du = \frac{dx}{\sqrt{1-x^2}} & v = \frac{x^3}{3} \end{array} \quad \begin{array}{l} \text{let } u = 1-x^2 \\ du = -2x \, dx \\ \text{so } x^2 = 1-u \end{array} \quad = \frac{x^3}{3} \arcsin x + \frac{1}{6} \int (u^{-1/2} - u^{1/2}) \, du$$

$$= \frac{x^3}{3} \arcsin x + \frac{1}{6} \left(2\sqrt{1-x^2} - \frac{2}{3} (1-x^2)^{3/2} \right) + C$$

$$2. D. \int \cos x \cos x \, dx = \cos x \sin x + \int \overset{=1-\cos^2 x}{\sin^2 x} \, dx = \cos x \sin x + \int dx - \int \cos^2 x \, dx$$

$$\begin{array}{ll} u = \cos x & dv = \cos x \, dx \\ du = -\sin x \, dx & v = \sin x \end{array}$$

$$\text{so } 2 \int \cos^2 x \, dx = \cos x \sin x + x + C$$

$$\text{so } \int \cos^2 x \, dx = \frac{1}{2} (\cos x \sin x + x) + C$$

3. $y = x^2 - 2x - 3 = (x-3)(x+1)$

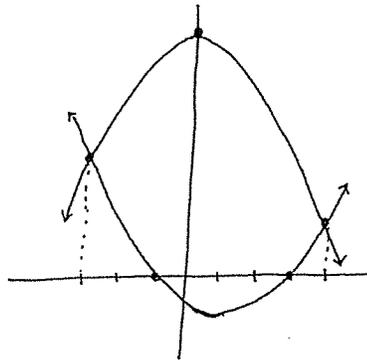
$y = 21 - x^2$

intersections: $x^2 - 2x - 3 = 21 - x^2$

$2x^2 - 2x - 24 = 0$

$2(x-4)(x+3) = 0$

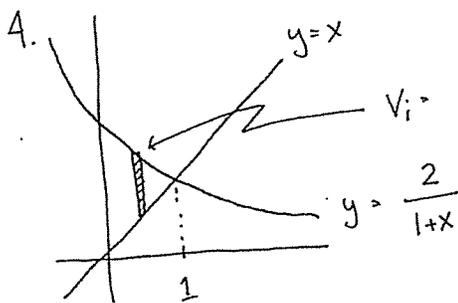
$x = 4$ $x = -3$



$$A_{\text{area}} = \int_{-3}^4 [(21 - x^2) - (x^2 - 2x - 3)] dx = \int_{-3}^4 (24 + 2x - 2x^2) dx = \left[24x + x^2 - \frac{2}{3}x^3 \right]_{-3}^4$$

$$= 24(4) + 16 - \frac{2}{3}(4)^3 - \left[24(-3) + 9 + 18 \right]$$

$$= \frac{343}{3}$$



$V_i = (\text{side})^2 \Delta x = \left(\frac{2}{1+x} - x \right)^2 \Delta x$

A. $\int_0^1 \left(\frac{2}{1+x} - x \right)^2 dx$

Added +1-1=0

B. $\int_0^1 \left(\frac{4}{(1+x)^2} - \frac{4x}{1+x} + x^2 \right) dx = \int_0^1 \frac{4}{(1+x)^2} dx - 4 \int_0^1 \frac{x+1-1}{1+x} dx + \frac{x^3}{3} \Big|_0^1$

$$= \frac{-4}{1+x} \Big|_0^1 - 4 \int_0^1 \left(1 - \frac{1}{1+x} \right) dx + \frac{1}{3} = \frac{-4}{2} + \frac{4}{1} - 4 \left(x - \ln|1+x| \right) \Big|_0^1 + \frac{1}{3}$$

$$= \frac{7}{3} - 4 \ln 2 = 4 \ln 2 - \frac{5}{3}$$

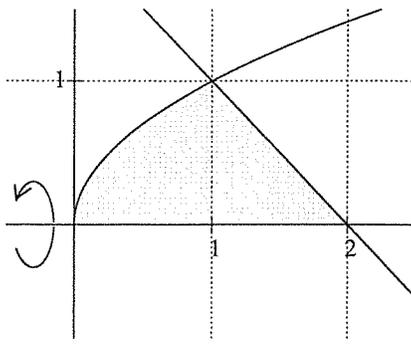


Figure 1:

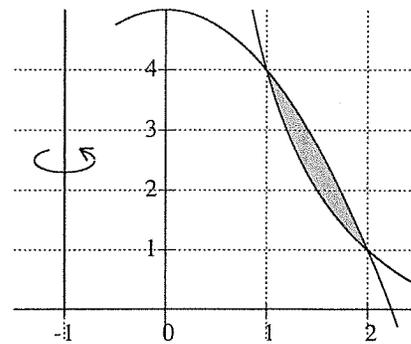


Figure 2:

$$y = \frac{6}{x} - 2 \text{ so } x = \frac{6}{y+2}$$

$$y = 5 - x^2$$

$$\text{so } x = \sqrt{5-y}$$

$$\begin{aligned}
 5. \text{ Average} &= \frac{3}{\pi} \int_0^{\pi/3} \frac{1}{1 - \sin x} dx = \frac{3}{\pi} \int_0^{\pi/3} \frac{1 + \sin x}{1 - \sin^2 x} dx = \frac{3}{\pi} \int_0^{\pi/3} \frac{1 + \sin x}{\cos^2 x} dx \\
 &= \frac{3}{\pi} \int_0^{\pi/3} (\sec^2 x + \tan x \sec x) dx = \frac{3}{\pi} \left[\tan x + \sec x \right]_0^{\pi/3} = \frac{3}{\pi} [\sqrt{3} + 2 - 1] \\
 &= \frac{3(\sqrt{3} + 1)}{\pi}
 \end{aligned}$$

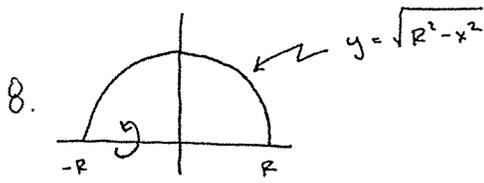
$$6. V_{\text{Disk}} = \pi \int_0^1 (\sqrt{x})^2 dx + \pi \int_1^2 (2-x)^2 dx$$

$$V_{\text{Shell}} = 2\pi \int_0^1 y(2-y-y^2) dy = 2\pi \left(y^2 - \frac{y^3}{3} - \frac{y^4}{4} \right) \Big|_0^1 = 2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}$$

$$7. V_{\text{Disk}} = \pi \int_1^4 \left[(\sqrt{5-y} + 1)^2 - \left(\frac{6}{y+2} + 1 \right)^2 \right] dy$$

$$V_{\text{Shell}} = 2\pi \int_1^2 (x+1) \left(5-x - \frac{6}{x} + 2 \right) dx = 2\pi \int_1^2 \left(1 + 7x - x^2 - x^3 - \frac{6}{x} \right) dx$$

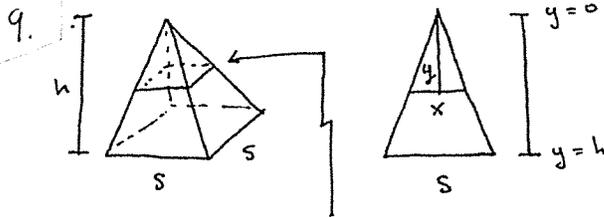
$$= 2\pi \left(x + \frac{7}{2}x^2 - \frac{x^3}{3} - \frac{x^4}{4} - 6 \ln|x| \right) \Big|_1^2 = 2\pi \left(\frac{65}{12} - 6 \ln 2 \right)$$



We use the Disk method to find the volume of the sphere formed by rotating the semicircle $y = \sqrt{R^2 - x^2}$ about the x-axis.

$$\text{Volume} = \pi \int_{-R}^R (\sqrt{R^2 - x^2})^2 dx = 2\pi \int_0^R (R^2 - x^2) dx = 2\pi \left(R^2x - \frac{x^3}{3} \right) \Big|_0^R = 2\pi \left(R^3 - \frac{R^3}{3} \right) = \frac{4}{3}\pi R^3$$

Symmetry ↕

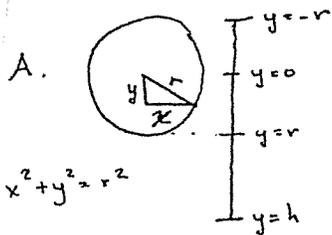


$$\frac{x}{y} = \frac{s}{h} \quad \text{so} \quad x = \frac{s}{h}y$$

$$W_i = fg (\text{side})^2 \Delta y (\text{distance}) = fg \left(\frac{s}{h}y \right)^2 (h-y) \Delta y$$

$$\begin{aligned} \text{Work} &= fg \frac{s^2}{h^2} \int_0^h (hy^2 - y^3) dy = fg \frac{s^2}{h^2} \left(\frac{hy^3}{3} - \frac{y^4}{4} \right) \Big|_0^h = fg \frac{s^2}{h^2} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) \\ &= fg \frac{s^2 h^2}{12} \end{aligned}$$

10.



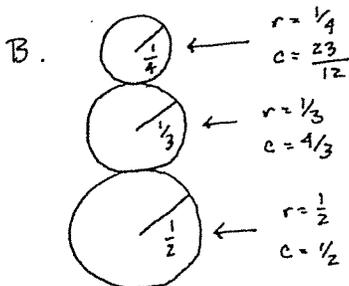
$$W_i = fg \pi x^2 \Delta y (h-y) = fg \pi (r^2 - y^2) (h-y) \Delta y$$

$$\text{Work} = fg \pi \int_{-r}^r (r^2 h - r^2 y - hy^2 + y^3) dy$$

$$= fg \pi \left[r^2 h y - \frac{r^2}{2} y^2 - \frac{h}{3} y^3 + \frac{y^4}{4} \right]_{-r}^r$$

$$= fg \pi \left[2r^3 h - \frac{2h}{3} r^3 \right] = \frac{4}{3} \pi r^3 h fg$$

Note: this is the distance the center is above the ground times the volume times f times g .



$$W = \frac{4}{3} \pi fg \left[\left(\frac{1}{2} \right)^3 \left(\frac{1}{2} \right) + \left(\frac{1}{3} \right)^3 \left(\frac{4}{3} \right) + \left(\frac{1}{4} \right)^3 \left(\frac{23}{12} \right) \right]$$