

Exam 2 Review - Solutions

$$1. A. \int_0^{\pi} \sin^3 5x \, dx = \int_0^{\pi} (1 - \cos^2 5x) \sin 5x \, dx = -\frac{1}{5} \int_{-1}^1 (1 - u^2) du = \frac{1}{5} \int_{-1}^1 (u^2 - 1) du$$

let $u = \cos 5x$

$$du = -5 \sin 5x \, dx \quad = -\frac{1}{5} \left(u - \frac{u^3}{3} \right) \Big|_{-1}^1 = \frac{1}{5} \left[\left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) \right] \leftarrow \text{Stop here}$$

$$x=0 \rightarrow u=1$$

$$x=\pi \rightarrow u=-1$$

$$= \frac{1}{5} \cdot \frac{4}{3} = \frac{4}{15}$$

$$B. \int \sin^9 x \cos^3 x \, dx = \int \sin^9 x (1 - \sin^2 x) \cos x \, dx = \int u^9 (1 - u^2) du$$

let $u = \sin x$

$$du = \cos x \, dx$$

$$= \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{12} + C$$

$$C. \int \sin^2 \pi x \, dx = \int \frac{1}{2} (1 - \cos 2\pi x) \, dx = \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) + C$$

$$D. \int \tan^2 x \sec^2 x \, dx = \frac{1}{3} \tan^3 x + C$$

$$u = \tan x \quad du = \sec^2 x \, dx$$

$$E. \int \tan^6 x \sec^4 x \, dx = \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx = \int u^6 (1 + u^2) du$$

$$u = \tan x \quad du = \sec^2 x \, dx$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

$$F. \int \sec^3 2x \tan^3 2x \, dx = \int \sec^2 2x (\sec^2 2x - 1) \sec 2x \tan 2x \, dx$$

$$u = \sec 2x \quad du = 2 \sec 2x \tan 2x \, dx$$

$$= \frac{1}{2} \int u^2 (u^2 - 1) \, du = \frac{1}{2} \int (u^4 - u^2) \, du$$

$$= \frac{\sec^5 2x}{10} - \frac{\sec^3 2x}{6} + C$$

2. A. $\int \sqrt{4-x^2} dx = \int 2\cos\theta \cdot 2\cos\theta d\theta = \int 2(1 + \cos 2\theta) d\theta$

Given:
 $\sin 2\theta = 2\sin\theta \cos\theta$

Let $x = 2\sin\theta$
 $dx = 2\cos\theta d\theta$

 $= 2(\theta + \frac{1}{2}\sin 2\theta) + C = 2(\theta + \sin\theta \cos\theta) + C$
 $= 2\left(2\cos\theta \sin\left(\frac{\theta}{2}\right) + \frac{x\sqrt{4-x^2}}{4}\right) + C$

B. $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x} dx = \int_0^{\pi/6} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta} \sqrt{3}\sec\theta \tan\theta d\theta = \int_0^{\pi/6} \tan^2\theta d\theta$

Let $x = \sqrt{3}\sec\theta$
 $dx = \sqrt{3}\sec\theta \tan\theta d\theta$

$x = \sqrt{3} \rightarrow \theta = 0$

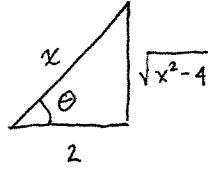
$x = 2 \rightarrow \theta = \pi/6$

$= \sqrt{3} (\tan\theta - \theta) \Big|_0^{\pi/6} = \sqrt{3} \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$

C. $\int \frac{x^2}{\sqrt{x^2-4}} dx = \int \frac{4\sec^2\theta}{2\tan\theta} \cdot 2\sec\theta \tan\theta d\theta = 4 \int \sec^3\theta d\theta$ This is a given integral

Let $x = 2\sec\theta$

$dx = 2\sec\theta \tan\theta d\theta$

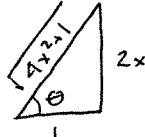


$= 2(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|) + C$
 $= 2\left(\frac{x}{2} \cdot \frac{\sqrt{x^2-4}}{2} + \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right|\right) + C$

D. $\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2} \int \frac{\sec^2\theta d\theta}{\sec\theta} = \frac{1}{2} \int \sec\theta d\theta$ given
 $= \frac{1}{2} \ln|\sec\theta + \tan\theta| + C$

Let $2x = \tan\theta$

$2dx = \sec^2\theta d\theta$

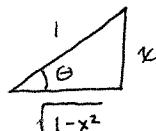


$= \frac{1}{2} \ln|\sqrt{4x^2+1} + 2x| + C$

E. $\int \frac{x^3}{\sqrt{1-x^2}} dx = \int \frac{\sin^3\theta}{\cos\theta} \cdot \cos\theta d\theta = \int (1-\cos^2\theta) \sin\theta d\theta = - \int (1-u^2) du$

Let $x = \sin\theta$

$dx = \cos\theta d\theta$



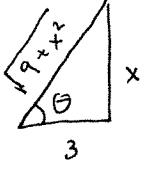
$u = \cos\theta$
 $du = -\sin\theta d\theta$

$= \frac{u^3}{3} - u + C = \frac{\cos^3\theta}{3} - \cos\theta + C$

$= \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + C$

$$2.F. \int \frac{9}{(9+x^2)^2} dx = \int \frac{9 \cdot 3 \sec^2 \theta}{9 \cdot 9 \sec^2 \theta} d\theta = \frac{1}{3} \int \frac{d\theta}{\sec^2 \theta} = \frac{1}{3} \int \cos^2 \theta d\theta$$

Let $x = 3 \tan \theta$
 $dx = 3 \sec^2 \theta d\theta$



$$= \frac{1}{6} \int (1 + \cos 2\theta) d\theta = \frac{1}{6} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{1}{6} (\theta + \sin \theta \cos \theta) + C \quad \text{given } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= \frac{1}{6} \left(\arctan \left(\frac{x}{3} \right) + \frac{3x}{9+x^2} \right) + C$$

$$3.A. \int \frac{3-x}{1-x^2} dx = \int \left(\frac{1}{1-x} + \frac{2}{1+x} \right) dx$$

$$= -\ln |1-x| + 2 \ln |1+x| + C$$

$$\frac{3-x}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$3-x = A(1+x) + B(1-x)$$

$$\text{Let } x=1, \quad 2=A(2) \text{ so } A=1$$

$$\text{Let } x=-1, \quad 4=B(-2) \text{ so } B=2$$

$$3. \int_{-1}^0 \frac{3}{x^2+x-2} dx = \int_{-1}^0 \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx = \left. \ln|x-1| - \ln|x+2| \right|_{-1}^0$$

$$= \ln|-1| - \ln|2| - \ln|-2| + \ln|1|$$

$$\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$= -2 \ln(2)$$

$$3 = A(x+2) + B(x-1)$$

$$\text{Let } x=1, \quad 3=A(3) \text{ so } A=1$$

$$\text{Let } x=-2, \quad 3=B(-3) \text{ so } B=-1$$

$$3.C. \int \frac{3x^2 + x + 9}{x^3 + 9x} dx = \int \left(\frac{1}{x} + \frac{2x+1}{x^2+9} \right) dx = \int \left(\frac{1}{x} + \frac{2x}{x^2+9} + \frac{1}{x^2+9} \right) dx$$

$$\frac{3x^2 + x + 9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$= \ln|x| + \ln(x^2+9) + \frac{1}{3} \operatorname{arctan}\left(\frac{x}{3}\right) + C$$

$$3x^2 + x + 9 = A(x^2+9) + (Bx+C)x$$

$$\text{Let } x=0, \quad 9 = A(9) \Rightarrow A=1$$

$$\text{Eqn. coeff: } x^2: \quad 3 = A+B \Rightarrow B=2$$

$$x: \quad 1 = C$$

$$3.D. \int \frac{4x^2 + 5x + 3}{x(x+1)^2} dx = \int \left(\frac{3}{x} + \frac{1}{x+1} - \frac{2}{(x+1)^2} \right) dx$$

$$\frac{4x^2 + 5x + 3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= 3\ln|x| + \ln|x+1| + \frac{2}{x+1} + C$$

$$4x^2 + 5x + 3 = A(x+1)^2 + Bx(x+1) + Cx$$

$$\text{Let } x=0, \quad 3 = A$$

$$\text{Eqn. coeff: } x^2: \quad 4 = A+B \Rightarrow B=1$$

$$x: \quad 5 = 2A+B+C \Rightarrow C=-2$$

$$3.E. \int \frac{x^3 - 4x^2 + 10x}{x^2 - 4x + 8} dx = \int \left(x + \frac{2x}{x^2 - 4x + 8} \right) dx = \int \left(x + \frac{2x-4}{x^2 - 4x + 8} + \frac{4}{x^2 - 4x + 8} \right) dx$$

split so
 that we
 can do this
 u-sub
 on
 the
 middle
 part

$$\frac{x^3 - 4x^2 + 10x}{x^2 - 4x + 8} = \frac{x}{x^2 - 4x + 8} + \frac{2x-4}{x^2 - 4x + 8} + \frac{4}{(x-2)^2 + 2^2}$$

$$u = x^2 - 4x + 8$$

$$du = (2x-4)dx$$

$$= \frac{x^2}{2} + \ln|x^2 - 4x + 8| + 2 \operatorname{arctan}\left(\frac{x-2}{2}\right) + C$$

$$F. \int \frac{\sin x}{\cos x - \cos^2 x} dx = - \int \frac{du}{u-u^2} = \int \frac{du}{u^2-u} = \int \left(\frac{1}{u-1} - \frac{1}{u} \right) du = \ln|u-1| - \ln|u| + C$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$\frac{1}{u^2-u} = \frac{A}{u-1} + \frac{B}{u}$$

$$1 = A(u-1) + B(u)$$

$$\text{let } u=0, 1=B(-1) \Rightarrow B=-1$$

$$u=1, 1=A \Rightarrow A=1$$

$$4.A. \int_{-\infty}^0 x e^x dx = x e^x \Big|_{-\infty}^0 - \int_{-\infty}^0 e^x dx = (x e^x - e^x) \Big|_{-\infty}^0 = \lim_{R \rightarrow \infty} (x e^x - e^x) \Big|_R^0$$

$$u=x \quad dv = e^x dx$$

$$du = dx \quad v = e^x$$

$$= \lim_{R \rightarrow \infty} (0 - 1 - R e^R + e^R)$$

$$= -1 - \lim_{R \rightarrow \infty} (R e^R)$$

← $(-\infty)(\infty)$ form
which is indeterminate,
so we use L'Hopital's Rule

$$= -1 - \lim_{R \rightarrow \infty} \left(\frac{R}{e^R} \right)$$

$\frac{(-\infty)}{\infty}$ form, so L'H applies

$$\stackrel{L'H}{=} -1 - \lim_{R \rightarrow \infty} \left(\frac{1}{-e^{-R}} \right) = -1$$

$$B. \int_4^\infty \frac{5}{x^2-x-6} dx = \int_4^\infty \left(\frac{1}{x-3} - \frac{1}{x+2} \right) dx = \lim_{R \rightarrow \infty} \left(\ln|x-3| - \ln|x+2| \right) \Big|_4^R$$

$$\frac{5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$= \lim_{R \rightarrow \infty} \left(\underbrace{\ln|R-3| - \ln|R+2|}_{\infty - \infty \text{ indeterminate form}} - \ln|4-3| + \ln|4+2| \right)$$

$$5 = A(x+2) + B(x-3)$$

$$\text{Let } x=3, 5=5A \Rightarrow A=1$$

$$x=-2, 5=-5B \Rightarrow B=-1$$

$$= \lim_{R \rightarrow \infty} \ln \left| \frac{R-3}{R+2} \right| + \ln 6$$

$$= \ln \left(\lim_{R \rightarrow \infty} \frac{R-3}{R+2} \right) + \ln 6 = \ln 6$$

↑ Since $\ln(\cdot)$ is continuous, the limit is the function can switch.

$$4.C. \int_0^{\infty} \frac{2}{3x+5} dx = \lim_{R \rightarrow \infty} \left[\frac{2}{3} \ln |3x+5| \right]_0^R = \lim_{R \rightarrow \infty} \frac{2}{3} \ln |3R+5| - \frac{2}{3} \ln 5 = \infty$$

so the integral diverges.

$$D. \int_0^{\infty} \frac{\arctan x}{x^2+1} dx = \lim_{R \rightarrow \infty} \left[\frac{1}{2} \arctan^2 x \right]_0^{\frac{\pi}{2}R} = \lim_{R \rightarrow \infty} \frac{1}{2} \arctan^2 R - 0 = \frac{1}{2} \left(\frac{\pi}{2} \right)^2 = \frac{\pi^2}{8}$$

$$E. \int_0^1 \ln x dx = x \ln x \Big|_0^1 - \int_0^1 dx = \lim_{R \rightarrow 0^+} (x \ln x - x) \Big|_R^1 = \lim_{R \rightarrow 0^+} (-1 - R \ln R + R)$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$= -1 - \lim_{R \rightarrow 0^+} (R \ln R) \leftarrow \infty \cdot (-\infty) \text{ indeterminate form}$$

$$= -1 - \lim_{R \rightarrow 0^+} \left(\frac{\ln R}{1/R} \right) \leftarrow \frac{-\infty}{\infty} \text{ indeterminate form, L'H applies}$$

$$\stackrel{\text{L'H}}{=} -1 - \lim_{R \rightarrow 0^+} \left(\frac{1/R}{-1/R^2} \right) = -1 - \lim_{R \rightarrow 0^+} (-R) = -1$$

$$F. \int_{-5}^2 \frac{dx}{\sqrt[5]{x-2}} = \int_{-7}^0 u^{-1/5} du = \lim_{R \rightarrow 0^-} \left[\frac{5}{4} u^{4/5} \right]_{-7}^R = \lim_{R \rightarrow 0^-} \left(\frac{5}{4} (R)^{4/5} - \frac{5}{4} (-7)^{4/5} \right)$$

$$\text{let } u = x-2$$

$$du = dx$$

$$= -\frac{5}{4} (-7)^{4/5}$$

$$x = -5 \mapsto u = -7$$

$$x \rightarrow 2 \mapsto u \rightarrow 0$$

$$5. A. \int_1^{\infty} \frac{dx}{x^4+x+7}$$

We have $0 \leq \frac{1}{x^4+x+7} \leq \frac{1}{x^4}$ for $x \geq 1$.

Additionally, the p-integral $\int_1^{\infty} \frac{dx}{x^4}$ converges since $p=4 > 1$.

So, by the Comparison Theorem, $\int_1^{\infty} \frac{dx}{x^4+x+7}$ also converges.

$$B. \int_2^{\infty} \frac{2}{x-\sqrt{x}} dx$$

Since $0 \leq \frac{2}{x} \leq \frac{2}{x-\sqrt{x}}$ for $x \geq 2$ & $\int_2^{\infty} \frac{2}{x} dx = 2 \int_2^{\infty} \frac{dx}{x}$ is a divergent

p-integral ($p=1 \leq 1$), by comparison, $\int_2^{\infty} \frac{2}{x-\sqrt{x}} dx$ also diverges.

C. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converges by comparison since $0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ for $x \in [0, \frac{\pi}{2}]$
 and $\int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is a convergent p-integral ($p=\frac{1}{2} < 1$)

Note: in the three arguments above, I wrote them in different orders. The important thing is that each contained three parts.

1) A comparison between functions (not integrals), e.g. $0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ for $x \in (0, \frac{\pi}{2}]$

2) Justification that an integral (not function) converges/diverges, \hookrightarrow This will almost always reference a p-integral.
 e.g. $\int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is a convergent p-integral ($p=\frac{1}{2} < 1$)

3) A conclusion about an integral (not a function)

e.g. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converges by comparison.

5.D. $\int_0^2 \frac{e^{x^2}}{x^2} dx$

Initially note that for $x \in (0, 2]$ $0 \leq \frac{1}{x^2} \leq \frac{e^{x^2}}{x^2}$.

Additionally, the p-integral $\int_0^2 \frac{dx}{x^2}$ diverges ($p=2 > 1$).

So, by the Comparison Theorem $\int_0^2 \frac{e^{x^2}}{x^2} dx$ also diverges.

E. $\int_2^\infty \frac{e^{-x^2}}{x^2+1} dx$

Since $e^{-x^2} \leq 1$ for all x , we have $0 \leq \frac{e^{-x^2}}{x^2+1} \leq \frac{1}{x^2}$.

Now, the p-integral $\int_2^\infty \frac{dx}{x^2}$ converges since $p=2 > 1$,

so by the Comparison Theorem $\int_2^\infty \frac{e^{-x^2}}{x^2+1} dx$ also converges.

F. $\int_0^2 \frac{dx}{x^2+\sqrt{x}}$

For $x \in (0, 2]$ we have $0 \leq \frac{1}{x^2+\sqrt{x}} \leq \frac{1}{\sqrt{x}}$. Since $\int_0^2 \frac{dx}{\sqrt{x}}$ is a convergent p-integral ($p=\frac{1}{2} < 1$), by the Comparison Theorem $\int_0^2 \frac{dx}{x^2+\sqrt{x}}$ also converges.

6. A. $p(x) = Cx^2$ on $[0, 3]$

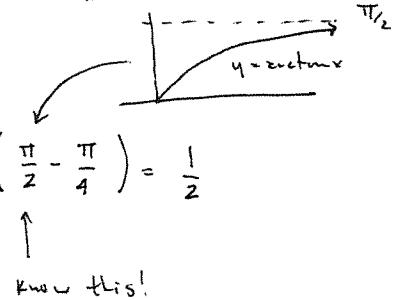
$$1 = \int_0^3 Cx^2 dx = \frac{C}{3} x^3 \Big|_0^3 = 9C \Rightarrow C = \frac{1}{9}$$

$$P(1 \leq X \leq 2) = \int_1^2 \frac{1}{9} x^2 dx = \frac{1}{27} x^3 \Big|_1^2 = \frac{7}{27}$$

B. $p(x) = \frac{C}{x^2+1}$ on $[0, \infty)$

$$1 = \int_0^\infty \frac{C}{x^2+1} dx = C \lim_{R \rightarrow \infty} (\arctan R - \arctan 0) = C \left(\frac{\pi}{2}\right) \Rightarrow C = \frac{2}{\pi}$$

$$P(X \geq 1) = \int_1^\infty \frac{2/\pi}{x^2+1} dx = \frac{2}{\pi} \lim_{R \rightarrow \infty} (\arctan R - \arctan 1) = \frac{2}{\pi} \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{2}$$



C. $p(x) = \frac{C}{\sqrt{4-x^2}}$ on $[0, 2]$

$$\begin{aligned} 1 &= \int_0^2 \frac{C}{\sqrt{4-x^2}} dx = C \cdot \lim_{R \rightarrow 2^-} \arcsin\left(\frac{x}{2}\right) \Big|_0^R = C \lim_{R \rightarrow 2^-} \arcsin\left(\frac{R}{2}\right) - \arcsin(0) \\ &= C \arcsin(1) = C \left(\frac{\pi}{2}\right) \Rightarrow C = \frac{2}{\pi} \end{aligned}$$

$$\begin{aligned} P(0 \leq X \leq 1) &= \int_0^1 \frac{2/\pi}{\sqrt{4-x^2}} dx = \frac{2}{\pi} \arcsin\left(\frac{x}{2}\right) \Big|_0^1 = \frac{2}{\pi} \left(\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right) \\ &= \frac{2}{\pi} \left(\frac{\pi}{6}\right) = \frac{1}{3} \end{aligned}$$

$$7.A. \quad y = 1 + 3x \quad [0, 2]$$

10

$$y' = 3$$

$$1 + (y')^2 = 10$$

$$S = \int_0^2 \sqrt{10} dx = 2\sqrt{10}$$

$$B. \quad y = \frac{1}{3}(2+x^2)^{\frac{3}{2}} \quad [0, 2]$$

$$y' = x(2+x^2)^{\frac{1}{2}}$$

$$1 + (y')^2 = 1 + 2x^2 + x^4$$

$$= (1+x^2)^2$$

$$S = \int_0^2 \sqrt{(1+x^2)^2} dx = \int_0^2 (1+x^2) dx = x + \frac{x^3}{3} \Big|_0^2 = 2 + \frac{8}{3} = \frac{14}{3}$$

$$C. \quad y = 1+x^2 \quad [0, 2]$$

$$y' = 2x$$

$$1 + (y')^2 = 1 + 4x^2$$

$$\text{let } 2x = \tan \theta$$

$$2dx = \sec^2 \theta d\theta$$

$$x=0 \rightarrow \theta = 0$$

$$x=2 \rightarrow \theta = \arctan 4$$

$$= \varphi \quad \sec \varphi = \sqrt{17}$$

$$\begin{array}{l} \text{given} \\ \text{triangle with height 4, base 1} \\ \text{angle } \varphi \text{ at bottom left} \\ \text{area} = \frac{1}{2} (4\sqrt{17} + \ln |\sqrt{17} + 4|) \end{array}$$

$$\text{Use } \Delta \text{ to unwrapp} \quad \sec(\arctan 4)$$

give the angle a name so we don't have to keep writing $\arctan 4$

$$D. \quad y = \frac{x^4}{32} + \frac{1}{x^2} \quad [1, 2]$$

$$y' = \frac{x^3}{8} - \frac{2}{x^3}$$

$$1 + (y')^2 = 1 + \left(\frac{x^6}{64} - \frac{1}{2} + \frac{4}{x^6} \right)$$

$$= \frac{x^6}{64} + \frac{1}{2} + \frac{4}{x^6}$$

$$= \left(\frac{x^3}{8} + \frac{2}{x^3} \right)^2$$

$$S = \int_1^2 \sqrt{\left(\frac{x^3}{8} + \frac{2}{x^3} \right)^2} dx = \int_1^2 \left(\frac{1}{8}x^3 + 2x^{-3} \right) dx$$

$$= \frac{x^4}{32} - \frac{1}{x^2} \Big|_1^2$$

$$= \left(\frac{16}{32} - \frac{1}{4} \right) - \left(\frac{1}{32} - 1 \right)$$

$$7.6. \quad y = \ln \cos x \quad [0, \frac{\pi}{3}]$$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$1 + (y')^2 = 1 + \tan^2 x \\ = \sec^2 x$$

$$S = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x} dx = \int_0^{\frac{\pi}{3}} \sec x dx$$

graph

$$= \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{3}}$$

$$= \ln |2 + \sqrt{3}| - \ln |1 + 0|$$

$$= \ln |2 + \sqrt{3}|$$

$$F. \quad y = \frac{x^5}{20} + \frac{1}{3x^3} \quad [1, 2]$$

$$y' = \frac{x^4}{4} - \frac{1}{x^4}$$

$$1 + (y')^2 = 1 + \left(\frac{x^8}{16} - \frac{1}{2} + \frac{1}{x^8} \right)$$

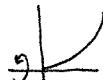
$$= \frac{x^8}{16} + \frac{1}{2} + \frac{1}{x^8} = \left(\frac{x^4}{4} + \frac{1}{x^4} \right)^2$$

$$S = \int_1^2 \sqrt{\left(\frac{x^4}{4} + x^{-4} \right)^2} dx = \int_1^2 \left(\frac{1}{4} x^4 + x^{-4} \right) dx$$

$$= \frac{x^5}{20} - \frac{1}{3x^3} \Big|_1^2$$

$$= \left(\frac{32}{20} - \frac{1}{24} \right) - \left(\frac{1}{20} - \frac{1}{3} \right)$$

$$8. A. \quad y = x^3 \quad [0, 1]$$



$$y' = 3x^2$$

$$1 + (y')^2 = 1 + 9x^4$$

$$S = 2\pi \int_0^1 x^3 \sqrt{1+9x^4} dx = 2\pi \cdot \frac{1}{36} \int_1^{10} u^{1/2} du$$

$$u = 1 + 9x^4$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

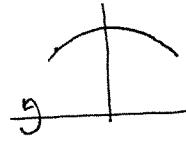
$$du = 36x^3 dx$$

$$x=0 \mapsto u=1$$

$$= \frac{\pi}{27} (10^{3/2} - 1)$$

$$x=1 \mapsto u=10$$

8.B. $y = \sqrt{4-x^2}$ $[-1, 1]$



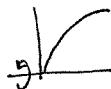
$$y' = \frac{-x}{\sqrt{4-x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{4-x^2}$$

$$= \frac{4-x^2+x^2}{4-x^2} = \frac{4}{4-x^2}$$

$$\int = 2\pi \int_{-1}^1 \sqrt{4-y^2} \sqrt{\frac{4}{4-y^2}} dy = 2\pi \int_{-1}^1 2 dy = 4\pi x \Big|_{-1}^1 = 8\pi$$

C. $y = \sqrt{x}$ $[0, 1]$



$$y' = \frac{1}{2\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{4x}$$

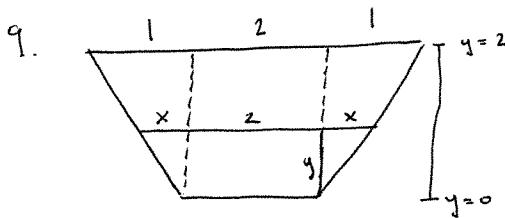
$$= \frac{4x+1}{4x}$$

$$\int = 2\pi \int_0^1 \sqrt{x} \sqrt{\frac{4x+1}{4x}} dx = \pi \int_0^1 \sqrt{4x+1} dx = \frac{\pi}{4} \int_1^5 u^{1/2} du$$

$$\begin{aligned} u &= 4x+1 \\ du &= 4 dx \\ x=0 &\mapsto u=1 \\ x=1 &\mapsto u=5 \end{aligned}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5$$

$$= \frac{\pi}{6} (5^{3/2} - 1)$$



By similar triangles $\frac{x}{y} = \frac{1}{2}$ so $x = \frac{1}{2}y$.

The width of a slice is $w = 2 + 2x = 2 + y$.

The area of a slice is $(2+y)\Delta y$.

The depth of a slice is $(2-y)$.

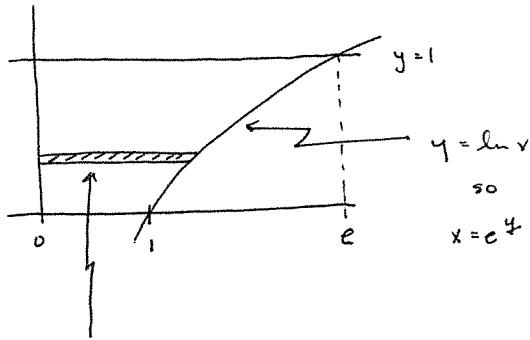
$$\text{Force} = \rho g \int_0^2 (2+y)(2-y) dy$$

$$= \rho g \int_0^2 (4-y^2) dy$$

$$= \rho g \left(4y - \frac{y^3}{3} \right) \Big|_0^2$$

$$= \frac{16}{3} \rho g$$

10.



The area of this slice is $x \Delta y = e^y \Delta y$.

The depth of this slice is $(1-y)$.

$$\text{Force} = \rho g \int_0^1 (1-y) e^y dy$$

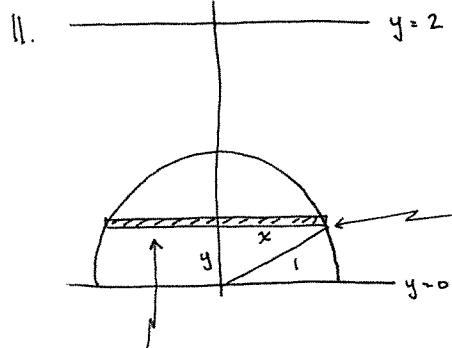
$$u = (1-y) \quad dv = e^y dy$$

$$du = -dy \quad v = e^y$$

$$= \rho g \left[(1-y) e^y \right]_0^1 + \int_0^1 e^y dy$$

$$= \rho g \left(0 - 1 + e^y \right)_0^1$$

$$= \rho g (0 - 1 + e - 1) = \rho g (e - 2)$$



The area of this slice is $2x \Delta y = 2\sqrt{1-y^2} \Delta y$.

The depth of this slice is $(2-y)$.

This is the quarter circle of radius 1, no need to do 2nd sub.

$$\text{Force} = \rho g \int_0^1 2(2-y)\sqrt{1-y^2} dy = 4\rho g \int_0^1 \sqrt{1-y^2} dy - 2\rho g \int_0^1 y\sqrt{1-y^2} dy$$

$$\begin{aligned} u &= 1-y^2 \\ du &= -2y dy \\ y=0 &\mapsto u=1 \\ y=1 &\mapsto u=0 \end{aligned}$$

$$= 4\rho g \left(\frac{\pi}{4} \right) + \rho g \int_1^0 u^{1/2} du$$

$$= 4\rho g \left(\frac{\pi}{4} \right) + \rho g \cdot \frac{2}{3} u^{3/2} \Big|_1^0$$

$$= \rho g \pi - \frac{2}{3} \rho g = \rho g \left(\pi - \frac{2}{3} \right)$$

14

12. A. $\int \cos^5 3x \sin^2 3x dx = \int (1 - \sin^2 3x)^2 \sin^2 3x \cos 3x dx = \frac{1}{3} \int (1 - 2u^2 + u^4) u^2 du$

let $u = \sin 3x$

$du = 3 \cos 3x dx$

$$= \frac{1}{3} \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C = \frac{\sin^3 3x}{9} - \frac{2 \sin^5 3x}{15} + \frac{\sin^7 3x}{21} + C$$

B. $\int \frac{x^3 + x^2 - 9x - 19}{x^2 - x - 6} dx = \int \left(x + 2 - \frac{x+7}{x^2-x-6} \right) dx = \int \left(x + 2 - \frac{2}{x-3} + \frac{1}{x+2} \right) dx$

$$\begin{array}{r} x+2 \\ x^2-x-6 \overline{) x^3+x^2-9x-19} \\ -x^3+x^2+6x \\ \hline 2x^2-3x-19 \\ -2x^2+2x+12 \\ \hline -x-7 \end{array}$$

$$\frac{x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = x^2 + 2x - 2 \ln|x-3| + \ln|x+2| + C$$

$x+7 = A(x+2) + B(x-3)$

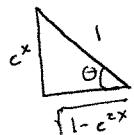
let $x=3$, $10 = A(5) \Rightarrow A=2$

let $x=-2$, $5 = B(-5) \Rightarrow B=-1$

C. $\int e^x \sqrt{1-e^{2x}} dx = \int \cos^2 \theta d\theta = \frac{1}{2} (\theta + \sin \theta \cos \theta) + C$

let $e^x = \sin \theta$

$e^x dx = \cos \theta d\theta$



$$= \frac{1}{2} (\arcsin(e^x) + e^x \sqrt{1-e^{2x}}) + C$$

D. $\int \sec^9 x \tan^3 x dx = \int \sec^8 x (\sec^2 x - 1) \sec x \tan x dx = \int (u^{10} - u^8) du$

$u = \sec x \quad du = \sec x \tan x dx$

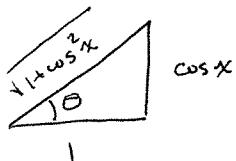
$$= \frac{1}{10} \sec^{10} x - \frac{1}{8} \sec^8 x + C$$

E. $\int \sin x \sqrt{1+\cos^2 x} dx = - \int \sec^3 \theta d\theta = -\frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C$

let $\cos x = \tan \theta$

$-\sin x dx = \sec^2 \theta d\theta$

$$= C - \frac{1}{2} \left(\cos x \sqrt{1+\cos^2 x} + \ln |\sqrt{1+\cos^2 x} + \cos x| \right) + C$$



1

$$12. F. \int \frac{6x^2 - 8x + 56}{x^4 - 16} dx$$

$$\frac{6x^2 - 8x + 56}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

$$6x^2 - 8x + 56 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x^2-4)$$

$$\text{Let } x = 2, \quad 24 - 16 + 56 = A(4)(8) \Rightarrow 64 = 32A \Rightarrow A = 2$$

$$\text{Let } x = -2, \quad 24 + 16 + 56 = B(-4)(8) \Rightarrow 96 = -32B \Rightarrow B = -3$$

$$\text{Eqn of coeff: } x^3: \quad 0 = A + B + C \Rightarrow C = 1$$

$$x^0: \quad 56 = 8A - 8B - 4D \Rightarrow D = -4$$

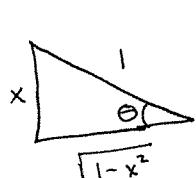
$$\text{so } \int \left(\frac{2}{x-2} - \frac{3}{x+2} + \frac{x}{x^2+4} - \frac{4}{x^2+4} \right) dx$$

$$= 2\ln|x-2| - 3\ln|x+2| + \frac{1}{2}\ln|x^2+4| - 2\arctan\left(\frac{x}{2}\right) + C$$

$$12. G. \int \sec^6 4x dx = \int (1 + \tan^2 4x)^2 \sec^2 4x dx = \frac{1}{4} \int (1 + 2u^2 + u^4) du$$

$$\begin{aligned} u &= \tan 4x \\ du &= 4 \sec^2 4x dx \end{aligned} \quad = \frac{1}{4} \left(\tan 4x + \frac{2}{3} \tan^3 4x + \frac{1}{5} \tan^5 4x \right) + C$$

$$H. \int \sqrt{\frac{1+x}{1-x}} dx = \int \sqrt{\frac{1-x^2}{(1-x)^2}} dx = \int \frac{\sqrt{1-x^2}}{(1-x)} dx = \int \frac{\cos^2 \theta d\theta}{1-\sin \theta} = \int \frac{1-\sin^2 \theta}{1-\sin \theta} d\theta$$



$$\begin{aligned} \text{let } x &= \sin \theta \\ dx &= \cos \theta d\theta \end{aligned}$$

$$= \int (1 + \sin \theta) d\theta$$

$$= \theta - \cos \theta + C$$

$$= \arcsin x - \sqrt{1-x^2} + C$$

$$\begin{aligned}
 12. i \int \cos^4 2x \, dx &= \frac{1}{4} \int (1 + \cos 4x)^2 \, dx = \frac{1}{4} \int (1 + 2\cos 4x + \cos^2 4x) \, dx \\
 &= \frac{1}{4} \int (1 + 2\cos 4x + \frac{1}{2} + \frac{1}{2} \cos 8x) \, dx = \frac{1}{4} \left(\frac{3}{2}x + \frac{1}{2} \sin 4x + \frac{1}{16} \sin 8x \right) + C
 \end{aligned}$$

$$j. \int \frac{5x^2 - 3x + 5}{x^3 + 5x} \, dx = \int \left(\frac{1}{x} + \frac{4x}{x^2 + 5} - \frac{3}{x^2 + 5} \right) \, dx$$

$$\frac{5x^2 - 3x + 5}{x(x^2 + 5)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 5} = \ln|x| + 2 \ln(x^2 + 5) - \frac{3}{15} \arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

$$5x^2 - 3x + 5 = A(x^2 + 5) + (Bx + C)x$$

$$\text{Let } x=0 : 5 = A(5) \Rightarrow A=1$$

$$\text{Eqnate Coeff : } x^2: 5 = A + B \Rightarrow B=4$$

$$x: -3 = C$$

$$\begin{aligned}
 k. \int \frac{\sqrt{x}}{x-4} \, dx &= \int \frac{u}{u^2-4} \cdot 2u \, du = \int \left(\frac{2u^2-8}{u^2-4} + \frac{8}{u^2-4} \right) \, du = \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2} \right) \, du \\
 &= 2\sqrt{x} + 2 \ln|\sqrt{x}-2| - 2 \ln|\sqrt{x}+2| + C
 \end{aligned}$$

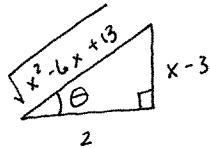
$$\begin{aligned}
 \text{Let } u &= \sqrt{x} & \frac{8}{u^2-4} &= \frac{A}{u-2} + \frac{B}{u+2} \\
 \text{so } u^2 &= x & 8 &= A(u+2) + B(u-2) \\
 2u \, du &= dx & u=2 : 8 &= 4A \Rightarrow A=2 \\
 & & u=-2 : 8 &= -4B \Rightarrow B=-2
 \end{aligned}$$

$$\begin{aligned}
 l. \int \sin^2 x \cos^2 x \, dx &= \frac{1}{4} \int (1 - \cos^2 2x) \, dx = \frac{1}{4} \int \left(1 - \frac{1}{2} - \frac{1}{2} \cos 4x \right) \, dx \\
 &= \frac{1}{4} \left(\frac{x}{2} - \frac{1}{8} \sin 4x \right) + C
 \end{aligned}$$

12. M. $\int \frac{2x}{(x^2 - 6x + 13)^2} dx = \underbrace{\int \frac{2x - 6}{(x^2 - 6x + 13)^2} dx}_{(1)} + \int \frac{6}{((x-3)^2 + 2^2)^2} dx$

For (1) use the substitution $u = x^2 - 6x + 13$, $du = (2x - 6) dx$ to get $-\frac{1}{x^2 - 6x + 13} + C$

For (2) Let $(x-3) = 2 \tan \theta$



$$dx = 2 \sec^2 \theta d\theta \quad \text{to get} \quad \int \frac{12 \sec^2 \theta}{16 \sec^4 \theta} d\theta = \frac{3}{4} \int \cos^2 \theta d\theta = \frac{3}{8} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{3}{8} (\theta + \sin \theta \cos \theta) + C = \frac{3}{8} \left(\arctan \left(\frac{x-3}{2} \right) + \frac{2(x-3)}{x^2 - 6x + 13} \right) + C$$

\uparrow
 $= \frac{1}{2} \sin 2\theta$

$$\text{so } \int \frac{2x}{(x^2 - 6x + 13)^2} dx = \frac{3}{8} \left(\arctan \left(\frac{x-3}{2} \right) + \frac{2(x-3)}{x^2 - 6x + 13} \right) - \frac{1}{x^2 - 6x + 13} + C$$

N. $\int_{-1}^4 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^4 \frac{dx}{x^2}$, both of which are divergent p-integrals, so $\int_{-1}^4 \frac{dx}{x^2}$ diverges.

O. $\int_{-1}^4 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 x^{-1/3} dx + \int_0^4 x^{-1/3} dx = \lim_{R \rightarrow 0^+} \left(\frac{3}{2} R^{2/3} - \frac{3}{2} (-1)^{2/3} \right) + \lim_{s \rightarrow 0^+} \left(\frac{3}{2} 4^{2/3} - \frac{3}{2} s^{2/3} \right) = \frac{3}{2} (4^{3/2} - 1)$

P. $\int \frac{3x^2 + 11x + 16}{(x+1)(x^2 + 6x + 13)} dx$

$$\frac{3x^2 + 11x + 16}{(x+1)(x^2 + 6x + 13)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 6x + 13}$$

$$\frac{3x^2 + 11x + 16}{(x+1)(x^2 + 6x + 13)} = A(x^2 + 6x + 13) + (Bx + C)(x+1)$$

~~Ex. 2x^2 + 11x + 16~~

$$\text{so } \int \frac{3x^2 + 11x + 16}{(x+1)(x^2 + 6x + 13)} dx = \int \left(\frac{1}{x+1} + \frac{2x+3}{x^2 + 6x + 13} \right) dx$$

$$= \ln|x+1| + \int \left(\frac{2x+6}{x^2 + 6x + 13} - \frac{3}{(x+3)^2 + 2^2} \right) dx$$

$$= \ln|x+1| + \ln(x^2 + 6x + 13) - \frac{3}{2} \arctan \left(\frac{x+3}{2} \right) + C$$

Let $x = -1$: $8 = A(-1)$ so $A = 1$

$$x^2 : 3 = A + B \quad \text{so } B = 2$$

$$x^0 : 16 = 13 + C \quad \text{so } C = 3$$

$$12. Q. \int \frac{e^x}{e^{2x} - e^x} dx = \int \frac{e^{-x}}{1 - e^{-x}} dx = \ln |1 - e^{-x}| + C$$

$$u = 1 - e^{-x}$$

$$du = e^{-x} dx$$

Note: Multiplying numerator & denominator by e^{-2x} is a trick I learned from Adrián Soto.

$$R. \int \frac{e^{2x}}{e^{2x} - e^x} dx = \int \frac{e^x}{e^x - 1} dx = \ln |e^x - 1| + C$$

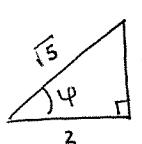
Much easier than $u = e^x$ &
using partial fractions.

$u = e^x - 1$
 $du = e^x dx$

$$S. \int_0^\varphi \sqrt{4+x^2} dx = \int_0^\varphi 2 \sec \theta \cdot 2 \sec^2 \theta d\theta = 2 \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Big|_0^\varphi$$

$$\text{Let } x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$



$$= 2 \left(\frac{\sqrt{5}}{2} \cdot \frac{1}{2} + \ln \left| \frac{\sqrt{5}}{2} + \frac{1}{2} \right| \right)$$

$$x=0 \rightarrow \theta=0$$

$$x=1 \rightarrow \theta = \arctan \left(\frac{1}{2} \right)$$

$\underbrace{\hspace{1cm}}$
 $= \varphi$

$$T. \int \sec x \tan^5 x dx = \int (\sec^2 x - 1)^2 \sec x \tan x dx = \int (u^4 - 2u^2 + 1) du$$

$u = \sec x$
 $du = \sec x \tan x dx$

$$= \frac{1}{5} \sec^5 x - \frac{2}{3} \sec^3 x + \sec x + C$$

$$U. \int \frac{2x^2 + 11x + 11}{(x+1)(x+2)(x+3)} dx = \int \left(\frac{1}{x+1} + \frac{3}{x+2} - \frac{2}{x+3} \right) dx = \ln|x+1| + 3 \ln|x+2| - 2 \ln|x+3| + C$$

$$\frac{2x^2 + 11x + 11}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$2x^2 + 11x + 11 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$$\text{Let } x = -1 : 2 = A(-1) \Rightarrow A = 1$$

Eqn'te coeff

$$x = -2 : -3 = B(-1) \Rightarrow B = 3$$

$$\begin{aligned} x^2 &: 2 = A + B + C \\ &\Rightarrow C = -2 \end{aligned}$$

$$| 3. A \cdot \int_0^{\infty} \frac{dx}{x^2 + 1} x = \int_0^1 \frac{dx}{x^2 + 1} + \int_1^{\infty} \frac{dx}{x^2 + 1} \quad (2)$$

we consider the "tell" part first, (1).

Since $0 < \frac{1}{x^2 + \sqrt{x}} < \frac{1}{\sqrt{x}}$ for $x \in (0, 1]$ & $\int_0^1 \frac{dx}{\sqrt{x}}$ is a convergent p-integral ($p = \frac{1}{2} < 1$),

By comparison, $\int_0^1 \frac{dx}{x^2 + \sqrt{x}}$ also converges.

Now consider the "long" part, (2).

Since $0 < \frac{1}{x^2+5x} < \frac{1}{x^2}$ for $x \in [1, \infty)$ & $\int_1^\infty \frac{dx}{x^2}$ is a convergent p-integral ($p=2 > 1$),

by comparison, $\int_{1}^{\infty} \frac{dx}{x^2 + \sqrt{x}}$ also converges.

Since (1) & (2) both converge, $\int_0^\infty \frac{dx}{x^2 + 1^x}$ also converges.

B. $\int_{\pi}^{\infty} \frac{|\sin x|}{x^2+1} dx$ converges by comparison since $0 < \frac{|\sin x|}{x^2+1} < \frac{1}{x^2}$ for $x > \pi$

$\int_{\pi}^{\infty} \frac{dx}{x^2}$ is a convergent p-integral ($p=2 > 1$).

$$C. \int_{-\infty}^{\infty} e^{-x^2} dx = \underbrace{\int_{-\infty}^{-1} e^{-x^2} dx}_{(1)} + \underbrace{\int_{-1}^{1} e^{-x^2} dx}_{(2)} + \underbrace{\int_{1}^{\infty} e^{-x^2} dx}_{(3)}$$

Since e^{-x^2} is continuous on $[-1,1]$, integral (2) is a proper integral & converges.

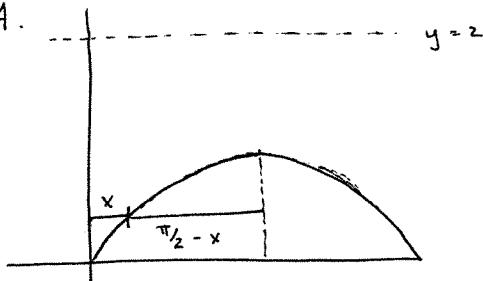
By symmetry (1) & (3) either both converge or both diverge, we consider only (3).

Since $0 < \frac{1}{e^{x^2}} < \frac{1}{x^2}$ & $\int_1^\infty \frac{dx}{x^2}$ is a convergent p-integral ($p=2 > 1$),

by comparison, integral (3) also converges. It follows that (1) does as well.

Since (1), (2) & (3) all converge, $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges. To calc it you will show $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

14.



LU //

The width at height y is $2\left(\frac{\pi}{2} - x\right) = \pi - 2\arcsin y$

$$\text{so force} = \rho g \int_0^1 (2-y)(\pi - 2\arcsin y) dy$$

$$u = \pi - 2\arcsin y \quad dv = (2-y) dy$$

$$du = \frac{-2}{\sqrt{1-y^2}} dy \quad v = 2y - \frac{y^2}{2}$$

$$= \rho g \left[\left(\pi - 2\arcsin y \right) \left(2y - \frac{y^2}{2} \right) \right]_0^1 + \int_0^1 \frac{4y - y^2}{\sqrt{1-y^2}} dy$$

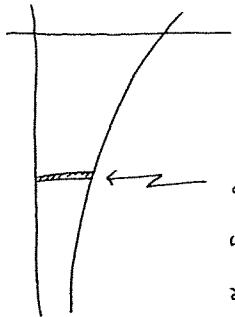
$$\begin{aligned} & y = \sin \theta & 0 & \mapsto 0 \\ & dy = \cos \theta d\theta & 1 & \mapsto \frac{\pi}{2} \\ & = \rho g \left[(\pi - 2 \cdot \frac{\pi}{2})(2 - \frac{1}{2}) - (\pi)(0) \right. \\ & \quad \left. + \int_0^{\frac{\pi}{2}} \frac{4 \sin \theta - \sin^2 \theta}{\cos \theta} \cancel{\cos \theta} d\theta \right] \end{aligned}$$

$$= \rho g \left(-4 \cos \theta - \frac{1}{2} (\theta - \frac{1}{2} \sin 2\theta) \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \rho g \left(4 - \frac{1}{2} \left(\frac{\pi}{2} \right) \right)$$

$$= \rho g \left(4 - \frac{\pi}{4} \right) //$$

15.

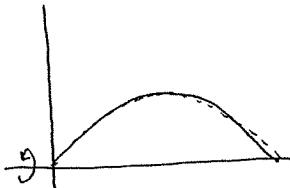


$$\begin{aligned} \text{depth} &= -y \\ \text{width} &= x = e^y \\ dx &= e^y dy \end{aligned}$$

$$\begin{aligned} \text{Force} &= fg \int_{-\infty}^0 -ye^y dy = fg(-1) \int_{-\infty}^0 ye^y dy \xrightarrow{\text{see \#4.A.}} \\ &= fg(-1)(-1) = fg \end{aligned}$$

21

16.



$$\begin{aligned} y &= \sin x \\ (1+y')^2 &= 1+\cos^2 x \end{aligned}$$

$$S = 2\pi \int_0^\pi \sin x \sqrt{1+\cos^2 x} dx \xrightarrow{\text{see \#12.E.}}$$

$$= 2\pi \left(-\frac{1}{2} \right) \left(\cos x \sqrt{1+\cos^2 x} + \ln \left| \sqrt{1+\cos^2 x} + \cos x \right| \right) \Big|_0^\pi$$

$$\begin{aligned} &= -\pi \left[-\sqrt{2} + \ln \left| \sqrt{2}+1 \right| - \sqrt{2} - \ln \left| \sqrt{2}+1 \right| \right] \\ &= \pi \left(2\sqrt{2} - \ln(\sqrt{2}-1) + \ln(\sqrt{2}+1) \right) \end{aligned}$$