

Exam 3 Review

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$$1. A. \sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

$$\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} \Rightarrow 2 = A(n+1) + B(n-1)$$

$$n=1, 2=2A \text{ so } A=1$$

$$n=-1, 2=-2B \text{ so } B=-1$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$S_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$S = \lim_{n \rightarrow \infty} S_n = \frac{3}{2}$$

$$B. \sum_{n=2}^{\infty} \frac{2}{n^2-n}$$

$$\frac{2}{n(n-1)} = \frac{A}{n-1} + \frac{B}{n} \Rightarrow 2 = An + B(n-1)$$

$$n=0, 2=-B \text{ so } B=-2$$

$$n=1, 2=A \text{ so } A=2$$

$$= \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n} \right)$$

$$S_n = \left(2 - \cancel{1}\right) + \left(\cancel{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \dots + \left(\frac{2}{n-1} - \frac{2}{n}\right)$$

$$= 2 - \frac{2}{n}$$

$$S = \lim_{n \rightarrow \infty} S_n = 2$$

$$C. \sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n}} = \sum_{n=1}^{\infty} 2^2 \left(\frac{2}{9}\right)^n = \frac{\frac{8}{9}}{1-\frac{2}{9}} = \frac{8}{9} \cdot \frac{9}{7} = \frac{8}{7}$$

$$D. \sum_{n=0}^{\infty} \frac{2^{2n}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{4}{3}\right)^n \text{ diverges since } |r| = \frac{4}{3} > 1.$$

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l.e. $\frac{1}{q} + \frac{q}{q^2} + \frac{q^2}{q^3} + \dots = \frac{\frac{1}{q}}{1 - q} = 1$ in general, $\frac{\text{first term}}{1 - \text{ratio}}$ when $|\text{ratio}| < 1$

f. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \dots$ diverges since $|r| = \frac{3}{2} > 1$.

g. $(\ln 1 - \ln \frac{1}{2}) + (\ln \frac{1}{2} - \ln \frac{1}{3}) + (\ln \frac{1}{3} - \ln \frac{1}{4}) + \dots$

Let $S_n = (\ln 1 - \ln \frac{1}{2}) + (\ln \frac{1}{2} - \ln \frac{1}{3}) + \dots + (\ln \frac{1}{n} - \ln \frac{1}{n+1})$
 $= \ln 1 - \ln \frac{1}{n+1} = 0 + \ln(n+1)$

$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \ln(n+1) = \infty$.

2. A. $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Let $a_n = \frac{1}{n^2-1}$ & $b_n = \frac{1}{n^2}$. Since $a_n, b_n > 0$ we can apply the Limit Comparison Test. Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ both series behave the same. We know $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series $\xrightarrow{(p=2>1)}$, so

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1} \text{ also converges by the Limit Comparison Test.}$$

B. $\sum_{n=1}^{\infty} \frac{\sqrt{9n+3}}{n+2}$
LCT solution: Let $a_n = \frac{\sqrt{9n+3}}{n+2}$ & $b_n = \frac{1}{\sqrt{n}}$. Since $a_n, b_n > 0$ & $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 3$, the Limit Comparison Test implies that $\sum_{n=1}^{\infty} \frac{\sqrt{9n+3}}{n+2}$ diverges since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \leq 1$).

Direct Comparison solution:

Initially we note $0 < \frac{1}{\sqrt{n}} < \frac{3\sqrt{n}}{2n} = \frac{\sqrt{9n}}{n+2} < \frac{\sqrt{9n+3}}{n+2}$ for $n > 2$. Since $\sum \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \leq 1$), by the Comparison Test $\sum_{n=1}^{\infty} \frac{\sqrt{9n+3}}{n+2}$ also diverges.

2.c

$$\sum_{n=2}^{\infty} \frac{3n+7}{\sqrt[3]{n^5-1}}$$

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Since $0 < \frac{1}{n^{2/3}} = \frac{n}{\sqrt[3]{n^5}} < \frac{3n+7}{\sqrt[3]{n^5-1}}$ & $\sum_{n=2}^{\infty} \frac{1}{n^{2/3}}$ is a divergent p-series ($p=2/3 \leq 1$),

by comparison $\sum_{n=2}^{\infty} \frac{3n+7}{\sqrt[3]{n^5-1}}$ also diverges.

D.

$$\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n+n^2}}$$

$\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n+n^2}}$ converges by comparison to the convergent p-series $\sum_{n=1}^{\infty} \frac{\pi}{n^2} = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}$ ($p=2 > 1$)

Since $0 < \frac{\pi}{\sqrt{n+n^2}} < \frac{\pi}{n^2}$.

E. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

LCT solution:

Let $a_n = \sin\left(\frac{1}{n^2}\right)$ & $b_n = \frac{1}{n^2}$. Since $a_n, b_n > 0$, we can apply the Limit Comparison Test.

$$\begin{aligned} \text{To do so we compute } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} \stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{\cos\left(\frac{1}{x^2}\right) \cdot -\frac{2}{x^3}}{-\frac{2}{x^3}} \\ &\stackrel{0/0 \text{ form}}{=} \lim_{x \rightarrow \infty} (\cos\left(\frac{1}{x^2}\right)) = \cos 0 = 1 \end{aligned}$$

Since the limit is positive & finite, $\sum a_n$ & $\sum b_n$ behave the same. In this case

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2 > 1$), $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ also converges.

Direct Comparison solution:

Since $\sin x = x - \frac{x^3}{3!} + \dots$ satisfies the A.S.T., for $x > 0$, $\sin x < x$. In particular,

$0 < \sin\left(\frac{1}{n^2}\right) < \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2 > 1$). By comparison $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ also converges.

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2. F. $\sum_{n=1}^{\infty} \frac{\ln n}{(n+1)^3}$

We know that for any $\epsilon > 0$, $\ln n < n^\epsilon$ for sufficiently large n .

With that in mind we make the comparison

$$0 \leq \frac{\ln n}{(n+1)^3} < \frac{n}{(n+1)^3} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2 > 1$), by comparison

$$\sum_{n=1}^{\infty} \frac{\ln n}{(n+1)^3} \text{ also converges.}$$

3. A. $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Let $f(x) = x e^{-x^2}$. $f(x)$ is positive & continuous. $f'(x) = e^{-x^2}(1 - 2x^2) < 0$

for $x > \frac{1}{\sqrt{2}}$, so $f(x)$ is decreasing and the Integral Test applies. We now consider

the integral

$$\int_1^{\infty} x e^{-x^2} dx = \frac{1}{2} \int_1^{\infty} e^{-u} du = \lim_{R \rightarrow \infty} -\frac{1}{2} e^{-u} \Big|_1^R = \lim_{R \rightarrow \infty} \left(-\frac{1}{2} e^{-R} + \frac{1}{2e} \right) = \frac{1}{2e}$$

$$\begin{aligned} \text{Let } u &= x^2 \\ du &= 2x dx \end{aligned}$$

Since the integral converges, by the Integral Test, $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ also converges.

B. $\sum_{n=2}^{\infty} \frac{1}{n^3 \sqrt{\ln n}}$

Let $f(x) = \frac{1}{x^3 \sqrt{\ln x}} = (x^3 \sqrt{\ln x})^{-1}$. Since $f'(x) = -(x^3 \sqrt{\ln x})^{-2} (\sqrt{\ln x} + \frac{1}{3} (\ln x)^{-2/3}) < 0$ for $x > 1$,

f is positive, decreasing & continuous for $x > 1$. The Integral Test applies, so we consider

$$\int_2^{\infty} \frac{dx}{x(\ln x)^{1/3}} = \int_{\ln 2}^{\infty} \frac{du}{u^{1/3}}$$

which is a divergent p-integral ($p = 1/3 \leq 1$). Since the integral diverges,
 $u = \ln x$
 $du = \frac{1}{x} dx$

by the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n^3 \sqrt{\ln n}}$ also diverges.

$$4. A. \sum_{n=1}^{\infty} \frac{\sin n \cos 5n}{n^2 + 3}$$

Since $0 < \left| \frac{\sin n \cos 5n}{n^2 + 3} \right| < \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2 > 1$),

by comparison $\sum_{n=1}^{\infty} \left| \frac{\sin n \cos 5n}{n^2 + 3} \right|$ also converges. We conclude then that

$$\sum_{n=1}^{\infty} \frac{\sin n \cos 5n}{n^2 + 3} \text{ converges absolutely.}$$

$$B. \sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1} \quad \text{Note: } \cos(\pi n) = (-1)^n$$

Since $0 < \frac{1}{n} < \left| \frac{\cos(\pi n)}{n-1} \right|$ & $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent Harmonic series, by comparison

$\sum_{n=2}^{\infty} \left| \frac{\cos(\pi n)}{n-1} \right|$ also diverges. So, $\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$ does not converge absolutely.

To test for conditional convergence we use the Alternating Series Test.

$$\text{Since } a_n = \frac{1}{n-1} > 0, \quad a_n = \frac{1}{n-1} > \frac{1}{n} = a_{n+1} \quad \& \quad \lim_{n \rightarrow \infty} \frac{1}{n-1} = 0, \quad \sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$$

converges by the Alternating Series Test.

Since $\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n-1}$ converges, but not absolutely, it converges conditionally.

$$C. \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{n^2 - 2n}$$

We begin by testing for absolute convergence, i.e. we consider the series $\sum_{n=3}^{\infty} \left| \frac{(-1)^{n-1}}{n^2 - 2n} \right| = \sum_{n=3}^{\infty} \frac{1}{n^2 - 2n} = \sum_{n=3}^{\infty} \frac{1}{n(n-2)}$

We use the Limit Comparison Test to compare to $\sum_{n=3}^{\infty} \frac{1}{n^2} = \sum_{n=3}^{\infty} b_n$. Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$, $\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2 > 1$), both positive series converge.

Since $\sum_{n=3}^{\infty} \left| \frac{(-1)^{n-1}}{n^2 - 2n} \right|$ converges, by definition $\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{n^2 - 2n}$ converges absolutely.

$$4.D. \sum_{n=4}^{\infty} \frac{(-1)^n (n-1)}{(n+1)}$$

Since $\lim_{n \rightarrow \infty} \frac{(-1)^n (n-1)}{(n+1)}$ does not exist, by the Test for Divergence $\sum_{n=4}^{\infty} \frac{(-1)^n (n-1)}{n+1}$ diverges.

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$$E. \sum_{n=5}^{\infty} \left(\frac{2n+3}{5n-7} \right)^n$$

We apply the root test to see

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{2n+3}{5n-7} \right)^n \right|} = \lim_{n \rightarrow \infty} \left(\frac{2n+3}{5n-7} \right) = \frac{2}{5} < 1.$$

Since $L < 1$, by the Root Test $\sum_{n=5}^{\infty} \left(\frac{2n+3}{5n-7} \right)^n$ converges absolutely.

$$F. \sum_{n=6}^{\infty} \frac{2^n n^2}{(2n+5)!}$$

We apply the Ratio Test.

$$\begin{aligned} f &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} (n+1)^2}{(2n+7)!} \cdot \frac{(2n+5)!}{2^n n^2} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{(2n+5)!}{(2n+7)(2n+6)(2n+5)!} \\ &= \lim_{n \rightarrow \infty} 2 \left(1 + \frac{1}{n} \right)^2 \cdot \frac{1}{(2n+7)(2n+6)} = 0 \end{aligned}$$

Since $f < 1$, by the Ratio Test $\sum_{n=6}^{\infty} \frac{2^n n^2}{(2n+5)!}$ converges absolutely.

$$5. \cos \pi = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = 1 - \frac{\pi^2}{2} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots \text{ is an alternating series}$$

that satisfies the assumptions of the Alternating Series Test. We can apply

the error bound $|S - S_n| < a_{n+1}$, in particular

$$\left| \cos \pi - \left(1 - \frac{\pi^2}{2} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} \right) \right| < \frac{\pi^{10}}{10!}$$

$$6. A. \sum_{n=0}^{\infty} \frac{n^n}{n!} (x-e)^n$$

Applying the Ratio Test gives

$$f = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!} (x-e)^{n+1}}{\frac{n^n}{(n)!} (x-e)^n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)n!} \cdot \left| \frac{(x-e)^{n+1}}{(x-e)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} \cdot \frac{(n+1)}{(n+1)} \cdot |x-e| = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n |x-e| = \underbrace{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n}_{\text{Note: } \lim_{n \rightarrow \infty} \left(1 + \frac{c}{n} \right)^n = e^c} |x-e|$$

$$= e |x-e|$$

The series converges when $f < 1$, i.e. $e|x-e| < 1$

$$\text{or } |x-e| < \frac{1}{e}, \text{ so } R = \frac{1}{e}$$

$$B. \sum_{n=1}^{\infty} (2n-1)! (x+2)^n$$

Applying the Ratio Test gives

$$f = \lim_{n \rightarrow \infty} \left| \frac{(2n+1)! (x+2)^{n+1}}{(2n-1)! (x+2)^n} \right| = \lim_{n \rightarrow \infty} (2n+1)(2n) |x+2| = \begin{cases} \infty & \text{if } x \neq -2 \\ 0 & \text{if } x = -2 \end{cases}$$

The series converges when $f < 1$, i.e. only for $x = -2$, so $R = 0$.

$$C. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

One more time, we apply the Ratio Test to find

$$f = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{|x^2|}{(2n+3)(2n+2)} = 0$$

Since $f = 0 < 1$ for all x , $R = \infty$

$$7. A. \sum_{n=1}^{\infty} \frac{x^n}{n}$$

To find R we use the Root Test.

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{n} \right|} = |x| \quad \text{Note: } \sqrt[n]{n} \xrightarrow{n \rightarrow \infty} 1$$

$$\text{So } |x| < 1$$

Checking the endpoints gives

$$x = -1 : \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ is the } \underline{\text{convergent}} \text{ Alternating Harmonic Series}$$

$$x = 1 : \sum_{n=1}^{\infty} \frac{1}{n} \text{ is the } \underline{\text{divergent}} \text{ Harmonic Series.}$$

so the interval of convergence is $I = [-1, 1)$.

$$B. \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} (x+4)^{2n}$$

To find R we use the Root Test.

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n}{4^n} (x+4)^{2n} \right|} = \lim_{n \rightarrow \infty} \frac{(x+4)^2}{4} = \frac{(x+4)^2}{4}$$

$$\text{The series converges when } L < 1, \text{ i.e. } \frac{(x+4)^2}{4} < 1 \quad \text{so} \quad (x+4)^2 < 4$$

$$|x+4| < 2$$

Checking the endpoints gives

$$x = -2 : \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} (-2)^{2n} = \sum_{n=0}^{\infty} (-1)^n \text{ which diverges by the Test for Divergence}$$

$$x = -6 : \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} (-6)^{2n} = \sum_{n=0}^{\infty} (-1)^n \quad \text{---} \quad \text{---}$$

so the interval of convergence is $I = (-6, -2)$.

$$7.c \quad \sum_{n=1}^{\infty} \frac{n-1}{n 2^n} (x-2)^n$$

To find R we use the Root Test.

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n-1}{n 2^n} (x-2)^n \right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n-1}{n}} \cdot \frac{|x-2|}{2} = \frac{|x-2|}{2}$$

The series converges when $\frac{|x-2|}{2} < 1$, i.e. $|x-2| < 2$.

Checking the endpoints gives

$$x=0 : \sum_{n=1}^{\infty} \frac{n-1}{n 2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{n-1}{n} (-1)^n \text{ which diverges by the Test for Divergence.}$$

$$x=4 : \sum_{n=1}^{\infty} \frac{n-1}{n 2^n} (2)^n = \sum_{n=1}^{\infty} \frac{n-1}{n} \text{ which also diverges by the Test for Divergence.}$$

so, the interval of convergence is $I = (0, 4)$

$$D. \quad \sum_{n=0}^{\infty} n! (x+\pi)^n$$

To find R we use the Ratio Test.

$$f = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! (x+\pi)^{n+1}}{n! (x+\pi)^n} \right| = \lim_{n \rightarrow \infty} (n+1) |x+\pi| = \begin{cases} \infty & \text{if } x \neq -\pi \\ 0 & \text{if } x = -\pi \end{cases}$$

The series only converges when $x = -\pi$, i.e. the interval of convergence is the single point $\{-\pi\}$.

E. $I = (-\infty, \infty)$, see 6.c.

$$F. \quad \sum_{n=1}^{\infty} \frac{(6x-12)^n}{n^2}$$

To find R we again apply Root Test $L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(6x-12)^n}{n^2} \right|} = |6x-12|$ so the series converges

for $|x-2| < \frac{1}{6}$. For $x = \frac{11}{6}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$x = \frac{13}{6}$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. So $I = [\frac{11}{6}, \frac{13}{6}]$

$$8. A \quad f(x) = \frac{4x}{2+x} = \frac{2x}{1-\left(-\frac{x}{2}\right)} = 2x \cdot \frac{1}{1-\left(-\frac{x}{2}\right)} \quad \text{so}$$

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$$f(x) = 2x \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{n-1}} \quad \text{for } \left| -\frac{x}{2} \right| < 1, \text{ i.e. } |x| < 2$$

$$B. \quad g(x) = \frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

$$C. \quad h(x) = x e^{x^2} = x \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} \quad \text{for all } x$$

$$D. \quad k(x) = \frac{5}{1+x^2} = 5 \frac{1}{1-(-x^2)} = 5 \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} 5(-1)^n x^{2n} \quad \text{for } |x^2| < 1 \\ \text{i.e. } |x| < 1$$

$$9. A. \quad f(x) = (5-x)^{3/2}; \quad c = \underline{\underline{-1}}$$

<u>n</u>	$\frac{f^{(n)}(x)}{(5-x)^{3/2}}$	$\frac{f^{(n)}(\underline{\underline{-1}})}{8}$
0	$(5-x)^{3/2}$	8
1	$\frac{3}{2}(5-x)^{1/2}(-1)$	-3
2	$\frac{3}{2} \cdot \frac{1}{2}(5-x)^{-1/2}$	$\frac{3}{8}$
3	$\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{-1}{2}(5-x)^{-3/2}(-1)$	$\frac{3}{64}$

$$\text{so } f(x) = 8 - 3(x-1) + \frac{3}{16}(x-1)^2 + \frac{1}{128}(x-1)^3 + \dots$$

$$9. B. \quad g(x) = \sin 2x; \quad c = \frac{\pi}{4}$$

<u>n</u>	<u>$\frac{g^{(n)}(x)}{\sin 2x}$</u>	<u>$\frac{g^{(n)}(\frac{\pi}{4})}{ }$</u>
0		1
1	$2 \cos 2x$	0
2	$-4 \sin 2x$	-4
3	$-8 \cos 2x$	0
	\vdots	\vdots

$$\text{so } g(x) = 1 - \frac{4}{2!} (x - \frac{\pi}{4})^2 + \frac{16}{4!} (x - \frac{\pi}{4})^4 - \frac{64}{6!} (x - \frac{\pi}{4})^6 + \dots$$

$$10. A. \lim_{x \rightarrow 0} \frac{\sin x - x + \frac{x^3}{6}}{2x^5} = \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \dots \right) - x + \frac{x^3}{6}}{2x^5}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots}{2x^5}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{240} - \frac{x^2}{2 \cdot 7!} + \frac{x^4}{2 \cdot 9!} + \dots \right) = \frac{1}{240}$$

$$13. \lim_{x \rightarrow 0} \frac{x \cos x}{1 - e^x} = \lim_{x \rightarrow 0} \frac{x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)}{1 - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)} = \lim_{x \rightarrow 0} \frac{x - \frac{x^3}{2!} + \frac{x^5}{4!} - \dots}{-x - \frac{x^2}{2!} - \frac{x^3}{3!} - \dots}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}{-1 - \frac{x^2}{2!} - \frac{x^3}{3!} - \dots} = -1$$

II. A.

$$\begin{aligned} f(x) = \sin 2x + e^{x^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \\ &= \left(2x - \frac{8}{3!} x^3 + \dots \right) + \left(1 + x^2 + \frac{x^4}{2!} + \dots \right) \\ &= 1 + 2x + x^2 - \frac{8}{3} x^3 + \dots \end{aligned}$$

$$\begin{aligned} B. \quad g(x) = \frac{e^x}{1-x} &= e^x \cdot \frac{1}{1-x} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \right) \left(1 + x + x^2 + x^3 + \dots \right) \\ &= 1 + 2x + \left(x^2 + x^2 + \frac{x^2}{2} \right) + \left(x^3 + x^3 + \frac{x^3}{2} + \frac{x^3}{6} \right) + \dots \\ &= 1 + 2x + \frac{5}{2} x^2 + \frac{8}{3} x^3 + \dots \end{aligned}$$

D. SEE TEXT OR SERIES REVIEW SHEET

13. True/False.

- (a) T F : If $\lim_{n \rightarrow \infty} a_n = 0$, the sequence $\{a_n\}$ converges.
- (b) T / F : If $\lim_{n \rightarrow \infty} a_n = 0$, the series $\sum a_n$ converges. } $\lim_{n \rightarrow \infty} a_n = 0$ given no information
- (c) T F : If $\lim_{n \rightarrow \infty} a_n = 0$, the series $\sum a_n$ diverges.
- (d) T / F : If $\lim_{n \rightarrow \infty} b_n = 7$, the sequence $\{b_n\}$ diverges.
- (e) T / F : If $\lim_{n \rightarrow \infty} b_n = 7$, the series $\sum b_n$ converges. } Test for Divergence
- (f) T F : If $\lim_{n \rightarrow \infty} b_n = 7$, the series $\sum b_n$ diverges.
- (g) T / F : If $0 < c_n < \frac{1}{n}$, the series $\sum c_n$ diverges. $0 < \frac{1}{n^2} < \frac{1}{n} \Rightarrow \sum \frac{1}{n^2}$ converges
- (h) T / F : The Harmonic series $\sum \frac{1}{n}$ converges.
- (i) T F : The Alternating Harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges. use A.S.T.
- (j) T / F : If $\sum |p_n|$ converges, then $\sum p_n$ converges conditionally.
- (k) T / F : If $\sum |q_n|$ diverges, then $\sum q_n$ diverges.

\hookrightarrow see (h) & (i) above

$\sum |p_n|$ converges

is the definition of

$\sum p_n$ converging absolutely

* No work is expected for this type of problem.

14. For each of the following series determine if they converge or diverge and then choose a test that can be used to show that.

(a) $\sum \left(1 + \frac{1}{n}\right)^{-n^2}$

Converges / Diverges

by the Root Test / Divergence Test.

$$\sqrt[n]{\left(1 + \frac{1}{n}\right)^{-n^2}} = \left(1 + \frac{1}{n}\right)^{-n} \xrightarrow{n \rightarrow \infty} \frac{1}{e}$$

(b) $\sum \left(1 + \frac{1}{n}\right)^n$

Converges / Diverges

by the Root Test / Divergence Test.

$$\left(1 + \frac{1}{n}\right)^n \xrightarrow{n \rightarrow \infty} e \neq 0$$

(c) $\sum \frac{1}{n \ln n}$

Converges / Diverges

by the Ratio Test / Integral Test.

Similar to 3(b)

(d) $\sum \frac{n+1}{n^2+2}$

Converges / Diverges

by the Limit Comparison Test / Ratio Test.

Compare to $\sum \frac{1}{n}$

(e) $\sum \frac{(-1)^n}{n}$

Converges / Diverges

by the Integral Test / Alternating Series Test.

(f) $\sum (-n)^n$

Converges / Diverges

by the Divergence Test / Alternating Series Test.

15. Use a series to integrate.

$$(A) e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{so} \quad \int e^{x^2} dx = A + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$$

constant.

$$(B) \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \quad \text{so} \quad \int \frac{\sin x}{x} dx = A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

$$(C) \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \quad \text{so} \quad \int \sin x^2 dx = A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

16. None! The Divergence Test can never be used to show convergence.

17. Only (c). (A) requires the Integral Test, (B) is not positive.

18. Both (A) & (B). (C) diverges by the Divergence Test

19. None! The A.S.T. can not be used to show divergence.

20. Both (B) & (C). Rational functions like (A) are always inconclusive.

21. Both (B) & (C).

22. Match a series on the left with a statement on the right to make a true statement.

$$1 - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{10} - \frac{1}{27} + \frac{1}{28} - \dots \quad \underline{\text{C}}$$

a) has radius of convergence $R = 7$.

Note: C also applies $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ E
 if this was $n=0$, then F
 $\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \right)$ if this was n, then H

b) has radius of convergence $R = 0$.

c) is a convergent alternating series.

d) is a divergent alternating series.

e) is the Maclaurin series for $\cos x - 1$.

f) is the Maclaurin series for $\cos x$.

g) is the Maclaurin series for $\sin x$.

h) is the Maclaurin series for $-\sin x$.

i) is a convergent geometric series.

j) is a divergent geometric series.

$$\sum_{n=7}^{\infty} 13^{3-2n} = \sum_{n=7}^{\infty} 13^3 \left(\frac{1}{13^2}\right)^n \quad \underline{\text{I}}$$

$$\sum_{n=1}^{\infty} n^n (x-7)^n \quad \underline{\text{B}}$$

23. For the following determine if the conclusion is a Valid use of the Comparison Test or an Invalid use.

- (a) I : Since $0 < \frac{1}{n} < \frac{1}{n-1}$ and $\sum \frac{1}{n}$ diverges, by comparison $\sum \frac{1}{n-1}$ also diverges.
- (b) V / I : Since $0 < \frac{1}{n+1} < \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges, by comparison $\sum \frac{1}{n+1}$ also diverges.
- (c) V / I : Since $0 < \frac{1}{n^2} < \frac{1}{n^2-1}$ and $\sum \frac{1}{n^2}$ converges, by comparison $\sum \frac{1}{n^2-1}$ also converges.
- (d) V / I : Since $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, by comparison $\sum \frac{1}{n^2+1}$ also converges.

24. For the following determine if the conclusion is a Valid use of the Limit Comparison Test or an Invalid use.

- (a) V / I : Since $\frac{1}{n}$ and $\frac{2n}{\sqrt{5n^4 + 2n^2}}$ are both positive, $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{5n^4 + 2n^2}} / \frac{1}{n} = \frac{2}{\sqrt{5}}$, and $\sum \frac{1}{n}$ diverges, by the Limit Comparison Test $\sum \frac{2n}{\sqrt{5n^4 + 2n^2}}$ also diverges.
- (b) V / I : Since $\frac{1}{n}$ and $\frac{2n}{\sqrt{5n^3 + 2n^2}}$ are both positive, $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{5n^3 + 2n^2}} / \frac{1}{n} = \infty$, and $\sum \frac{1}{n}$ diverges, by the Limit Comparison Test $\sum \frac{2n}{\sqrt{5n^3 + 2n^2}}$ also diverges.
- (c) V / I : Since $\frac{1}{\sqrt{n}}$ and $\frac{2n}{\sqrt{5n^4 + 2n^2}}$ are both positive, $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{5n^4 + 2n^2}} / \frac{1}{\sqrt{n}} = 0$, and $\sum \frac{1}{\sqrt{n}}$ diverges, by the Limit Comparison Test $\sum \frac{2n}{\sqrt{5n^4 + 2n^2}}$ also diverges.

25. Consider the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$. Determine if the following are True or False.

- (a) T F : If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}$, then the series converges for $x = 0$. \leftarrow so $R = 4$
- (b) T / F : If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series converges for $x = 0$. \leftarrow so $R = 0$
- (c) T / F : If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 4$, then the series converges for $x = 0$. \leftarrow so $R = \frac{1}{4}$
- (d) T F : If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 0$, then the series converges for $x = 0$. \leftarrow so $R = \infty$

26. Consider the power series $\sum_{n=0}^{\infty} b_n(x+2)^n$. Assume the series converges for $x = 1$ and diverges for $x = -6$. For each of the following values of x determine if the series Converges, Diverges, or if there is Not enough information to tell.

- (a) C / D / N : $x = -5$
 (b) C / D / N : $x = 0$
- $3 \leq R \leq 4$
- (c) C / D / N : $x = 2$
 (d) C / D / N : $x = 4$

27. Assume that $f(x)$ has the following power series representation.

$$f(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \dots$$

What is $f^{(5)}(0)$? = $(5!)(8)$