

Math 172 Final Exam Review

1. A. $\int (x+3) \sin 2x \, dx = -\frac{(x+3) \cos 2x}{2} + \int \frac{\cos 2x}{2} \, dx = \frac{\sin 2x}{4} - \frac{(x+3) \cos 2x}{2} + C$

$u = x+3 \quad dv = \sin 2x \, dx$

$du = dx \quad v = -\frac{\cos 2x}{2}$

B. $\int \sin x \cos^7 x \, dx = -\int u^7 \, du = -\frac{\cos^8 x}{8} + C$

$u = \cos x$

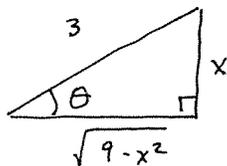
$du = -\sin x \, dx$

C. $\int \sqrt{9-x^2} \, dx = \int 9 \cos^2 \theta \, d\theta = \frac{9}{2} \int (1 + \cos 2\theta) \, d\theta = \frac{9}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C$

let $x = 3 \sin \theta$

$dx = 3 \cos \theta \, d\theta$

So $9-x^2 = 9-9\sin^2 \theta = 9\cos^2 \theta$



$= \frac{9}{2} (\theta + \sin \theta \cos \theta + C)$

$= \frac{9}{2} \left(\arcsin \left(\frac{x}{3} \right) + \frac{x \sqrt{9-x^2}}{9} \right) + C$

$\swarrow \sin 2\theta = 2 \sin \theta \cos \theta$

D. $\int \frac{7x-6}{(2x+1)(x+3)} \, dx = \int \left(\frac{1}{2x+1} + \frac{3}{x+3} \right) \, dx = \frac{1}{2} \ln |2x+1| + 3 \ln |x+3| + C$

$\frac{7x+6}{(2x+1)(x+3)} = \frac{A}{2x+1} + \frac{B}{x+3}$

$7x+6 = A(x+3) + B(2x+1)$

let $x = -3: -15 = B(-5) \Rightarrow B = 3$

$x = -\frac{1}{2}: \frac{5}{2} = A\left(\frac{5}{2}\right) \Rightarrow A = 1$

E. $\int x^2 e^{-4x} dx = -\frac{1}{4}x^2 e^{-4x} + \int \frac{1}{2}x e^{-4x} dx = -\frac{1}{4}x^2 e^{-4x} - \frac{1}{8}x e^{-4x} + \int \frac{1}{8} e^{-4x} dx$

$u = x^2 \quad dv = e^{-4x} dx$
 $du = 2x dx \quad v = -\frac{1}{4}e^{-4x}$

$u = \frac{1}{2}x \quad dv = e^{-4x} dx$
 $du = \frac{1}{2}dx \quad v = -\frac{1}{4}e^{-4x}$

$= -\frac{1}{4}x^2 e^{-4x} - \frac{1}{8}x e^{-4x} - \frac{1}{32}e^{-4x} + C$

F. $\int \sec^4 4\theta \tan^4 4\theta d\theta = \int (\tan^2 4\theta + 1) \tan^4 4\theta \sec^2 4\theta d\theta = \frac{1}{4} \int (u^2 + 1) u^4 du$

$\swarrow = \sec^2 4\theta$

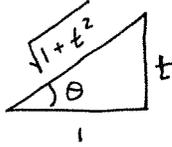
let $u = \tan 4\theta$
 $du = 4 \sec^2 4\theta d\theta$

$= \frac{1}{4} \left(\frac{\tan^7 4\theta}{7} + \frac{\tan^5 4\theta}{5} \right) + C$

G. $\int \frac{dt}{t^4 + 2t^2 + 1} = \int \frac{dt}{(t^2 + 1)^2} = \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta$

Let $t = \tan \theta$
 $dt = \sec^2 \theta d\theta$

so $t^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta$



$\swarrow \sin 2\theta = 2 \cos \theta \sin \theta$

$= \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C$

$= \frac{1}{2} \left(\theta + \sin \theta \cos \theta \right) + C$

$= \frac{1}{2} \left(\arctan t + \frac{t \cdot 1}{(\sqrt{1+t^2})^2} \right) + C$

$= \frac{1}{2} \left(\arctan t + \frac{t}{1+t^2} \right) + C$

H. $\int \frac{4x^2 + 3x + 2}{x^3 + x^2} dx = \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1} \right) dx$

$\frac{4x^2 + 3x + 2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \ln|x| - \frac{2}{x} + 3 \ln|x+1| + C$

$4x^2 + 3x + 2 = A(x+1) + B(x+1) + Cx^2$

Let $x=0$: $2 = B$ Equate coeff to find A

Let $x=-1$: $3 = C$ x^2 : $4 = A + C \Rightarrow A = 1$

I. $\int_1^e t \ln t \, dt = \frac{1}{2} t^2 \ln t \Big|_1^e - \int_1^e \frac{1}{2} t \, dt = \frac{1}{2} e^2 - \frac{1}{4} t^2 \Big|_1^e = \frac{1}{2} e^2 - \frac{1}{4} e^2 + \frac{1}{4} = \frac{1}{4} (e^2 + 1)$

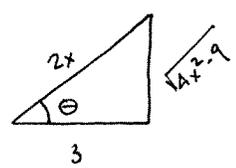
$u = \ln t \quad dv = t \, dt$
 $du = \frac{1}{t} \, dt \quad v = \frac{1}{2} t^2$

J. $\int \sin^2(\pi x) \cos^2(\pi x) \, dx = \frac{1}{4} \int \sin^2(2\pi x) \, dx = \frac{1}{8} \int (1 - \cos(4\pi x)) \, dx$
 $= \frac{1}{8} \left(x - \frac{1}{4\pi} \sin(4\pi x) \right) + C$

Trigonometric identities used:
 $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$
 $\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)$

K. $\int \frac{\sqrt{4x^2-9}}{x} \, dx = \int \frac{3 \tan \theta}{\frac{3}{2} \sec \theta} \cdot \frac{3}{2} \sec \theta \tan \theta \, d\theta = 3 \int \tan^2 \theta \, d\theta = 3 \int (\sec^2 \theta - 1) \, d\theta$

Let $2x = 3 \sec \theta$
 $2 \, dx = 3 \sec \theta \tan \theta \, d\theta$



So $4x^2 - 9 = 9 \sec^2 \theta - 9 = 9 \tan^2 \theta$

$= 3 \left(\tan \theta - \theta \right) + C$
 $= 3 \left(\frac{\sqrt{4x^2-9}}{3} - \operatorname{arcsec} \left(\frac{2x}{3} \right) \right) + C$

L. $\int \frac{5x+6}{(x-2)(x^2+4)} \, dx = \int \left(\frac{2}{x-2} - \frac{2x}{x^2+4} + \frac{1}{x^2+2^2} \right) \, dx$
 $u = x^2+4 \quad du = 2x \, dx$

$\frac{5x+6}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$
 $= 2 \ln|x-2| - \ln(x^2+4) + \frac{1}{2} \operatorname{arctan} \left(\frac{x}{2} \right) + C$

$5x+6 = A(x^2+4) + (Bx+C)(x-2)$

Let $x=2$: $16 = A(8) \Rightarrow A=2$

Equate coeff: x^2 x^0
 $0 = A+B$ $6 = 4A-2C$
 So $B=-2$ So $C=1$

Note: $\int \frac{du}{u^2+a^2} = \frac{1}{a} \operatorname{arctan} \left(\frac{u}{a} \right) + C$
 is given information.

$$M. \int (x + \sin x)^2 dx = \int (x^2 + 2x \sin x + \sin^2 x) dx = \frac{x^3}{3} + 2 \sin x - 2x \cos x + \frac{1}{2}x - \frac{1}{4} \sin 2x + C$$

Note:

$$2 \int x \sin x dx = -2x \cos x + 2 \int \cos x dx = -2x \cos x + 2 \sin x + C$$

$$u = x \quad dv = \sin x dx$$

$$du = dx \quad v = -\cos x$$

$$\int \sin^2 x dx = \frac{1}{2} \int (1 - \cos 2x) dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C$$

$$N. \int \frac{x^2}{x^2 + 4x + 13} dx = \int \left(\frac{x^2 + 4x + 13}{x^2 + 4x + 13} - \frac{4x + 13}{x^2 + 4x + 13} \right) dx = \int \left(1 - \frac{2(2x+4)}{x^2 + 4x + 13} - \frac{5}{(x+2)^2 + 3^2} \right) dx$$

$u = x^2 + 4x + 13$
 $du = (2x+4) dx$

$$= x - 2 \ln(x^2 + 4x + 13) - \frac{5}{3} \operatorname{arctan} \left(\frac{x+2}{3} \right) + C$$

$$O. \int \sin 2\theta e^{\sin \theta} d\theta = \int 2 \sin \theta \cos \theta e^{\sin \theta} d\theta = \int 2x e^x dx = 2x e^x - \int 2e^x dx$$

$$\text{Let } x = \sin \theta \quad u = 2x \quad dv = e^x dx$$

$$dx = \cos \theta d\theta \quad du = 2 dx \quad v = e^x$$

$$= 2x e^x - 2e^x + C$$

$$= 2e^{\sin \theta} (\sin \theta - 1) + C$$

$$P. \int \sec^2 x \sec^3(\tan x) \tan^3(\tan x) dx = \int \sec^3 \theta \tan^3 \theta d\theta = \int \sec^2 \theta (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta$$

$\tan^2 \theta$

$$\text{Let } \theta = \tan x \quad u = \sec \theta \quad du = \sec \theta \tan \theta d\theta$$

$$d\theta = \sec^2 x dx$$

$$= \int (u^4 - u^2) du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \sec^5(\tan x) - \frac{1}{3} \sec^3(\tan x) + C$$

$$Q. \int \frac{e^{2x}}{e^{2x} + 5e^x + 6} dx = \int \frac{u}{u^2 + 5u + 6} du = \int \left(\frac{3}{u+3} - \frac{2}{u+2} \right) du$$

$$\text{Let } u = e^x \\ du = e^x dx$$

$$\frac{u}{(u+3)(u+2)} = \frac{A}{u+3} + \frac{B}{u+2}$$

$$= 3 \ln(e^x + 3) - 2 \ln(e^x + 2) + C$$

$$u = A(u+2) + B(u+3)$$

$$\text{let } u = -2 : -2 = B$$

$$u = -3 : -3 = -A \Rightarrow A = 3$$

$$R. \int_0^{\pi/2} \cos x \sqrt{1 + 3 \sin^2 x} dx = \frac{1}{\sqrt{3}} \int_0^{\pi/3} \sec^3 \theta d\theta = \frac{1}{2\sqrt{3}} \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Big|_0^{\pi/3}$$

$$\text{Let } \tan \theta = \sqrt{3} \sin x$$

$$\sec^2 \theta d\theta = \sqrt{3} \cos x dx$$

$$= \frac{1}{2\sqrt{3}} \left(2 \cdot \sqrt{3} + \ln |2 + \sqrt{3}| \right)$$

$$\begin{array}{ccc} \frac{x}{0} & \longmapsto & \frac{\theta}{0} \\ \frac{\pi/2}{2} & & \frac{\pi/3}{3} \end{array}$$

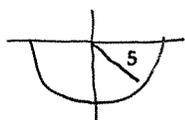
$$2. A \quad x = t + 1 \quad \text{so } x - 1 = t \quad \text{so } y = (x-1)^3 + 1$$

$$B. \quad x = e^{2t} + 1 \quad \text{so } x - 1 = e^{2t} \quad \text{so } y = (x-1)^3 + 1$$

$$C. \quad x = \frac{2+t}{1+t} = 1 + \frac{1}{1+t} \quad \text{so } x - 1 = \frac{1}{1+t} \quad \text{so } y = (x-1)^3 + 1$$

$$y = \frac{1}{(1+t)^3} + 1$$

$$D. \quad x = 5 \cos t \quad y = 5 \sin t \quad t \in [\pi, 2\pi]$$



$$\text{so } y = -\sqrt{25 - x^2}$$

3. A. $x = 4t + 1$
 $y = 3t - 1$
 $0 \leq t \leq 2$

$x' = 4$
 $y' = 3$

so $ds = \sqrt{(x')^2 + (y')^2} dt = 5 dt$

$S = \int_0^2 5 dt = 10$

B. $x = 3t^2 + 3$
 $y = t^3 - 2$
 $0 \leq t \leq \sqrt{5}$

$x' = 6t$
 $y' = 3t^2$

$(x')^2 + (y')^2 = 36t^2 + 9t^4$

$ds = \sqrt{36t^2 + 9t^4} dt$
 $= 3t\sqrt{4 + t^2} dt$

$S = \int_0^{\sqrt{5}} 3t\sqrt{4 + t^2} dt = \frac{3}{2} \int_4^9 u^{1/2} du$
 $u = 4 + t^2$
 $du = 2t dt$
 $= u^{3/2} \Big|_4^9 = 27 - 8$

$\frac{t}{0} \rightarrow \frac{u}{4}$
 $\frac{t}{\sqrt{5}} \rightarrow \frac{u}{9}$

$= 19$

C. $x = e^{2t} \cos t$
 $y = e^{2t} \sin t$
 $t \in [0, \pi]$

Sec 7.E.

← This is a cycloid.

D. $x = t - \sin t$
 $y = 1 - \cos t$
 $t \in [0, 2\pi]$

$x' = 1 - \cos t$
 $y' = \sin t$

$(x')^2 + (y')^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t$

$= 2 - 2\cos t$

$= 2(1 - \cos t) \leftarrow 1 - \cos t = 2 \sin^2 \left(\frac{t}{2}\right)$

$= 4 \sin^2 \frac{t}{2}$ so $ds = 2 \sin \frac{t}{2} dt$

$S = \int_0^{2\pi} ds = \int_0^{2\pi} 2 \sin \frac{t}{2} dt = -4 \cos \left(\frac{t}{2}\right) \Big|_0^{2\pi} = -4(-1 - 1) = 8$

4. A. $x = 2 \sin t$
 $y = t$

$x' = 2 \cos t$
 $y' = 1$

$x'(\pi/6) = \sqrt{3}$

$y'(\pi/6) = 1$

so $\frac{dy}{dx} \Big|_{t=\pi/6} = \frac{1}{\sqrt{3}}$ $\therefore \frac{ds}{dt} \Big|_{t=\pi/6} = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

B. $x = \frac{t+2}{4}$
 $y = \frac{t-3}{3}$

$x' = \frac{1}{4}$
 $y' = \frac{1}{3}$

so $\frac{dy}{dx} \Big|_{t=\frac{-\pi^3}{e^{37}}} = \frac{1/3}{1/4} = \frac{4}{3}$

$\therefore \frac{ds}{dt} \Big|_{t=\text{ugly number}} = \sqrt{\frac{1}{16} + \frac{1}{9}} = \frac{5}{12}$

4.c. $x = 2 \cos 2t$ $x' = -4 \sin 2t$ $x'(\frac{3\pi}{8}) = -4(\frac{\sqrt{2}}{2}) = -2\sqrt{2}$
 $y = \sin 2t$ $y' = 2 \cos 2t$ $y'(\frac{3\pi}{8}) = 2(-\frac{\sqrt{2}}{2}) = -\sqrt{2}$

so $\left. \frac{dy}{dx} \right|_{t=\frac{3\pi}{8}} = \frac{-\sqrt{2}}{-2\sqrt{2}} = \frac{1}{2}$ $\left. \frac{ds}{dt} \right|_{t=\frac{3\pi}{8}} = \sqrt{(-2\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{10}$

D. $x = \frac{t}{t+1}$ $x' = \frac{1}{(t+1)^2}$ $x'(1) = \frac{1}{4}$
 $y = t^2$ $y' = 2t$ $y'(1) = 2$ so $\left. \frac{dy}{dx} \right|_{t=1} = \frac{2}{\frac{1}{4}} = 8$ $\left. \frac{ds}{dt} \right|_{t=1} = \sqrt{\frac{1}{16} + 4} = \frac{\sqrt{65}}{4}$

5. A. $r = 2$ so $x^2 + y^2 = 4$

B. $r = \frac{3}{\cos \theta}$ so $r \cos \theta = 3$ or $x = 3$

C. $r = 2 \csc \theta$ so $r \sin \theta = 2$ or $y = 2$

D. $r = \frac{6}{2 \cos \theta - 3 \sin \theta}$ so $2r \cos \theta - 3r \sin \theta = 6$ or $2x - 3y = 6$

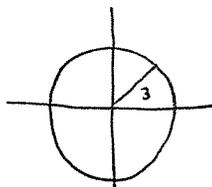
E. $r = -2 \sin \theta$ so $r^2 = -2r \sin \theta$ or $x^2 + y^2 = -2y$

or if you want to be fancy,

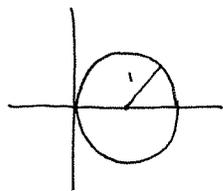
$$x^2 + (y+1)^2 = 1$$

F. $\theta = \frac{\pi}{4}$ or $y = x$

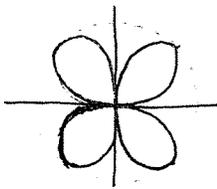
6. A. $r=3$



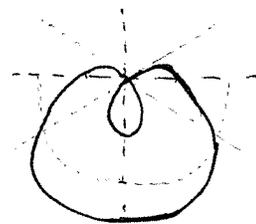
B. $r=2\cos\theta$



C. $r=\sin 2\theta$



D. $r=1-2\sin\theta$



See 9.c for the upside down version

7. A. $r=3$

$r'=0$

so $ds = \sqrt{9+0} d\theta = 3 d\theta$

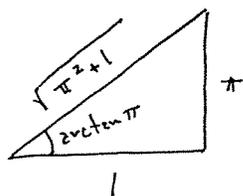
$S = \int_0^{2\pi} 3 d\theta = 6\pi$

B. $r=\theta$

$\frac{dr}{d\theta} = r' = 1$

so $ds = \sqrt{\theta^2 + 1} d\theta$

so $S = \int_0^{\pi} \sqrt{\theta^2 + 1} d\theta = \int_0^{\arctan \pi} \sec^3 \phi d\phi$



so $\sec(\arctan \pi) = \sqrt{\pi^2 + 1}$

Let $\theta = \tan \phi$
 $d\theta = \sec^2 \phi d\phi$

$\frac{\theta}{\pi} \mapsto \frac{\phi}{\arctan(\pi)}$

$= \frac{1}{2} \left(\sec \phi \tan \phi + \ln |\sec \phi + \tan \phi| \right) \Big|_0^{\arctan(\pi)}$
 $= \frac{1}{2} \left(\pi \sqrt{\pi^2 + 1} + \ln(\pi + \sqrt{\pi^2 + 1}) \right)$

C. $r=\theta^2$

$r'=2\theta$

so $ds = \sqrt{\theta^4 + 4\theta^2} d\theta$
 $= \theta \sqrt{\theta^2 + 4} d\theta$

so $S = \int_0^{\pi} \theta \sqrt{\theta^2 + 4} d\theta = \int_4^{\pi^2 + 4} \frac{1}{2} u^{1/2} du = \frac{1}{3} u^{3/2} \Big|_4^{\pi^2 + 4}$

$u = \theta^2 + 4$
 $du = 2\theta d\theta$

$= \frac{1}{3} \left[(\pi^2 + 4)^{3/2} - 8 \right]$

D. $r = \theta^2 - 1$

$r' = 2\theta$

so $ds = \sqrt{(\theta^2 - 1)^2 + (2\theta)^2} d\theta$
 $= \sqrt{\theta^4 - 2\theta^2 + 1 + 4\theta^2} d\theta$
 $= \sqrt{\theta^4 + 2\theta^2 + 1} d\theta$
 $= \sqrt{(\theta^2 + 1)^2} d\theta$

so $S = \int_0^{\pi} (\theta^2 + 1) d\theta = \frac{\theta^3}{3} + \theta \Big|_0^{\pi}$
 $= \frac{\pi^3}{3} + \pi$

7. E. $r = e^{2\theta}$
 $r' = 2e^{2\theta}$

so $ds = \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta$
 $= \sqrt{5}e^{2\theta} d\theta$

$S = \int_0^\pi \sqrt{5}e^{2\theta} d\theta = \frac{\sqrt{5}}{2} e^{2\theta} \Big|_0^\pi = \frac{\sqrt{5}}{2} (e^{2\pi} - 1)$

F. $r = \cos^2 \theta$

$r' = -2 \cos \theta \sin \theta$



$(r)^2 + (r')^2 = \cos^4 \theta + 4 \cos^2 \theta \sin^2 \theta$

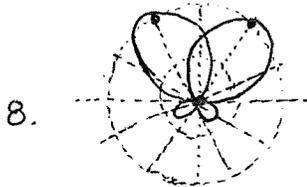
$= \cos^2 \theta (\cos^2 \theta + 4 \sin^2 \theta)$

$= \cos^2 \theta (1 + 3 \sin^2 \theta)$

$4 \sin^2 \theta = \sin^2 \theta + 3 \sin^2 \theta$

so $S = 4 \int_0^{\pi/2} \cos \theta \sqrt{1 + 3 \sin^2 \theta} d\theta = \frac{2}{\sqrt{3}} (2\sqrt{3} + \ln |2 + \sqrt{3}|)$ ← see #1.R

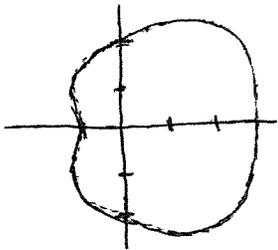
Symmetry



B.

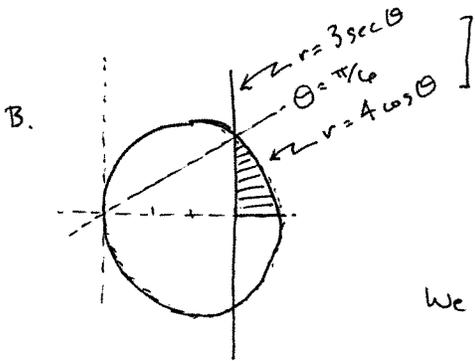
← A cute little butterfly! (or see next page for a fancy version)

9. A. Inside $r = 2 + \cos \theta$



$\frac{1}{2} \int_0^{2\pi} (2 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos \theta + \cos^2 \theta) d\theta = \frac{1}{2} (1 + \cos 2\theta)$

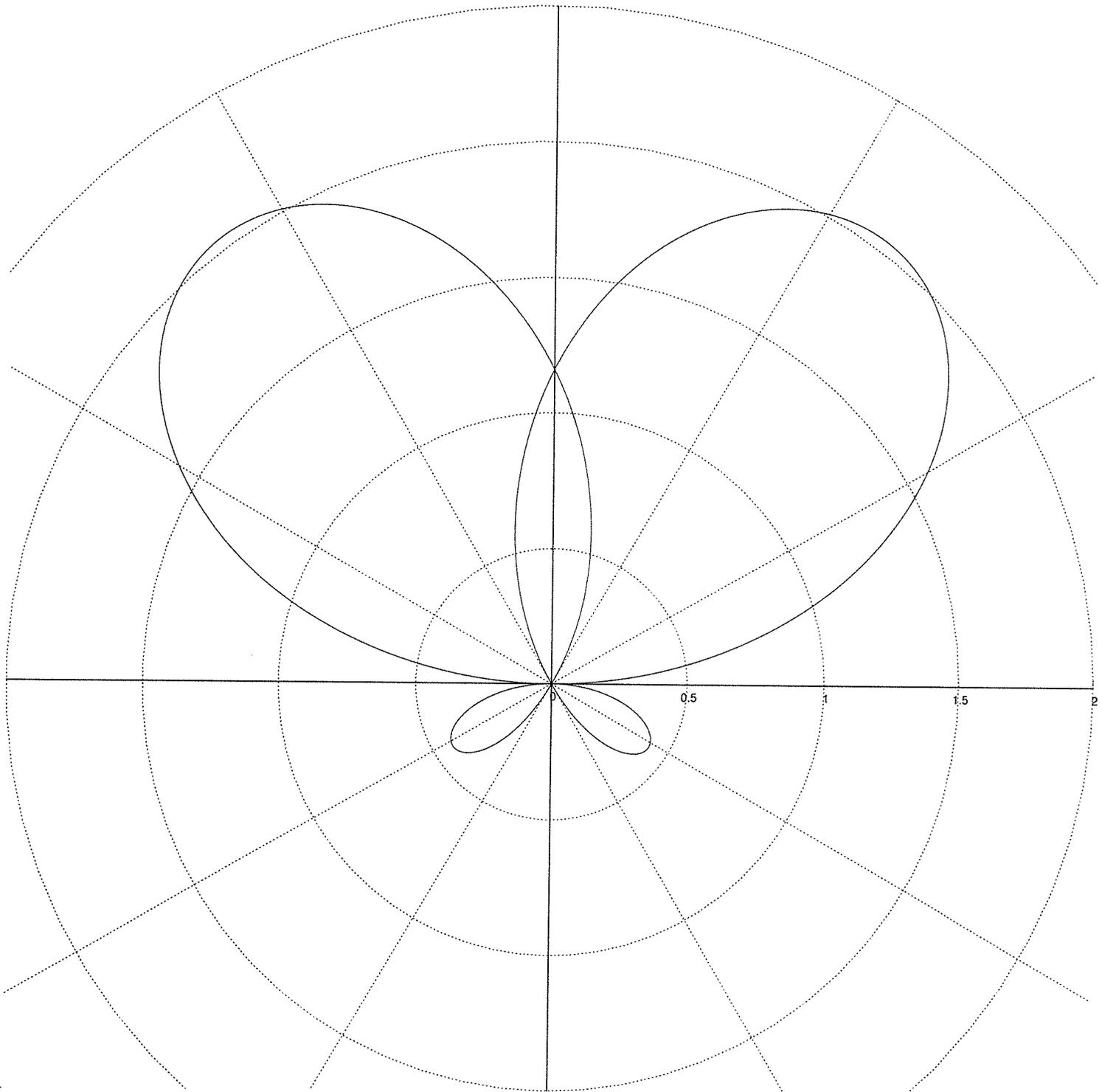
$= \frac{1}{2} (4\theta + 4 \sin \theta + \frac{\theta}{2} + \frac{1}{4} \sin 2\theta) \Big|_0^{2\pi} = \frac{9\pi}{2}$



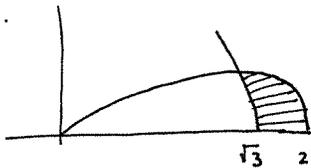
Intersect when $3 \sec \theta = 4 \cos \theta$
 $\frac{3}{4} = \cos^2 \theta$
 $\frac{\sqrt{3}}{2} = \cos \theta$
 so $\theta = \pi/6$

We consider only the top half, by symmetry

$2 \cdot \frac{1}{2} \int_0^{\pi/6} (4^2 \cos^2 \theta - 9 \sec^2 \theta) d\theta = \int_0^{\pi/6} (8 + 8 \cos 2\theta - 9 \sec^2 \theta) d\theta$
 $= 8\theta + 4 \sin 2\theta - 9 \tan \theta = \frac{4\pi}{3} + 4 \frac{\sqrt{3}}{2} - \frac{9}{\sqrt{3}} = \frac{4\pi}{3} - \sqrt{3}$



10.4.



$r = \sqrt{3}$ & $r = 2 \cos 3\theta$ intersect when $\frac{\sqrt{3}}{2} = \cos 3\theta$

$$\text{so } 3\theta = \frac{\pi}{6}$$

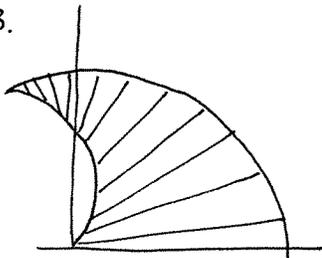
$$\text{so } \theta = \frac{\pi}{18}$$

Symmetry

$$6 \cdot \frac{1}{2} \int_0^{\pi/18} (4 \cos^2 3\theta - 3) d\theta = 3 \int_0^{\pi/18} (2 + 2 \cos 6\theta - 3) d\theta = 3 \left(-\theta + \frac{1}{3} \sin 6\theta \right) \Big|_0^{\pi/18}$$

$$= 3 \left(-\frac{\pi}{18} + \frac{1}{3} \cdot \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

13.



$r = 3$ & $r = 2 - 2 \cos \theta$ intersect when $-\frac{1}{2} = \cos \theta$

$$\text{so } \theta = \frac{2\pi}{3}$$

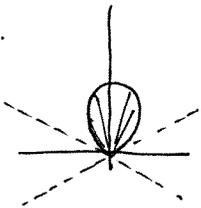
Symmetry

$$2 \cdot \frac{1}{2} \int_0^{2\pi/3} (9 - 4 + 8 \cos \theta - 4 \cos^2 \theta) d\theta$$

$$= \int_0^{2\pi/3} (5 + 8 \cos \theta - 2 - 2 \cos 2\theta) d\theta$$

$$= 3\theta + 8 \sin \theta - \sin 2\theta \Big|_0^{2\pi/3} = 2\pi + 8 \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 2\pi + \frac{9\sqrt{3}}{2}$$

14.

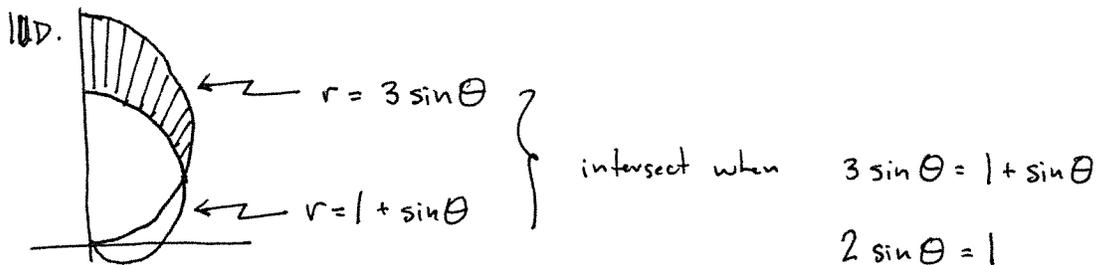


The inner loop happens when $r \leq 0$, so $1 + 2 \sin \theta \leq 0$ when $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$

$$\frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 2 \sin \theta)^2 d\theta = \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta = \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (3 + 4 \sin \theta - 2 \cos 2\theta) d\theta$$

$$= \frac{1}{2} (3\theta - 4 \cos \theta - \sin 2\theta) \Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} = \frac{1}{2} (2\pi - 4 \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right) - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right))$$

$$= \pi - \frac{3\sqrt{3}}{2}$$



$$2 \sin \theta = 1$$

$$\sin \theta = \frac{1}{2}$$

$$\text{so } \theta = \frac{\pi}{6}$$

Symmetry

$$2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} (9 \sin^2 \theta - (1 + 2 \sin \theta + \sin^2 \theta)) d\theta$$

$$= \int_{\pi/6}^{\pi/2} (8 \sin^2 \theta - 1 - 2 \sin \theta) d\theta$$

$$= \int_{\pi/6}^{\pi/2} (4 - 4 \cos 2\theta - 1 - 2 \sin \theta) d\theta$$

$$= \int_{\pi/6}^{\pi/2} (3 - 4 \cos 2\theta - 2 \sin \theta) d\theta = 3\theta - 2 \sin 2\theta + 2 \cos \theta \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{3\pi}{2} - \left(\frac{\pi}{2} - 2 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{\sqrt{3}}{2} \right)$$

$$= \pi //$$

Practice Final A

1. A. $\int \sin^3 2x \cos^2 2x dx = -\frac{1}{2} \int (1-u^2)u^2 du = \frac{1}{2} \int (u^4 - u^2) du = \frac{1}{2} \left(\frac{u^5}{5} - \frac{u^3}{3} \right) + C$

let $u = \cos 2x$
 $du = -2 \sin 2x dx$

$$= \frac{1}{2} \left(\frac{\cos^5 2x}{5} - \frac{\cos^3 2x}{3} \right) + C$$

1. B. $\int_1^2 x^2 \ln 3x dx = \left. \frac{x^3}{3} \ln 3x \right|_1^2 - \int_1^2 \frac{x^2}{3} dx = \left(\frac{8}{3} \ln 6 - \frac{1}{3} \ln 3 \right) - \left. \frac{x^3}{9} \right|_1^2$

$u = \ln 3x \quad dv = x^2 dx$
 $du = \frac{3}{3x} dx = \frac{1}{x} dx \quad v = \frac{x^3}{3}$

$$= \frac{8}{3} \ln 6 - \frac{1}{3} \ln 3 - \frac{7}{9}$$

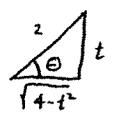
C. $\int \frac{4x}{x^2-1} dx = \int \left(\frac{2}{x-1} + \frac{2}{x+1} \right) dx = 2 \ln |x-1| + 2 \ln |x+1| + C$

$\frac{4x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$ ~ or ~
 $Ax = A(x+1) + B(x-1)$ $u = x^2 - 1$
 $du = 2x dx$ $\int \frac{2 du}{u} = 2 \ln |u| + C$
 $2 \ln |x^2 - 1| + C$

let $x=1 \Rightarrow A=2$
 $x=-1 \Rightarrow B=2$

D. $\int \frac{dt}{t^2 \sqrt{4-t^2}} = \int \frac{2 \cos \theta}{4 \sin^2 \theta (2 \cos \theta)} d\theta = \frac{1}{4} \int \csc^2 \theta d\theta = -\frac{1}{4} \cot \theta + C$

let $t = 2 \sin \theta$
 $dt = 2 \cos \theta d\theta$



$$= C - \frac{1}{4} \cdot \frac{\sqrt{4-t^2}}{t} = C - \frac{\sqrt{4-t^2}}{t}$$

2. $x = \frac{1}{\sqrt{t+1}} \quad y = \frac{t}{t+1}$

$x^2 = \frac{1}{t+1} \quad y = \frac{1-x^2}{1-x^2+x^2} = 1-x^2$ $y = 1-x^2$

$t+1 = \frac{1}{x^2}$

$t = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2}$

- or -
 $y = \frac{\frac{1-x^2}{x^2} - 1}{\frac{1}{x^2}} = 1-x^2$

3. $(2t, t+1)$ for $0 \leq t \leq 15$

$$x' = 6t \quad y' = 3t^2$$

$$(x')^2 = 36t^2 \quad (y')^2 = 9t^4$$

$$(x')^2 + (y')^2 = 36t^2 + 9t^4 = 9t^2(4+t^2)$$

$$s = \int_0^{\sqrt{5}} 3t \sqrt{4+t^2} dt = \frac{3}{2} \int_4^9 u^{1/2} du = u^{3/2} \Big|_4^9 = 27 - 8 = 19$$

$$u = 4+t^2 \quad 0 \rightarrow 4$$

$$du = 2t dt \quad \sqrt{5} \rightarrow 9$$

4. $(\tan t, 2t)$

$$x' = \sec^2 t \quad y' = 2$$

$$\text{Speed} \Big|_{t=\pi/4} = \sqrt{\sec^4 t + 4} \Big|_{t=\pi/4} = \sqrt{4+4} = 2\sqrt{2}$$

$\sec \frac{\pi}{4} = \sqrt{2}$

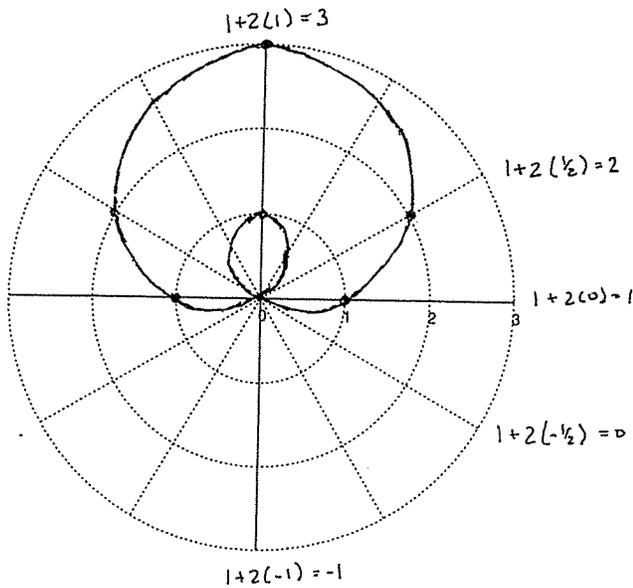
5. $\frac{dy}{dx} = \frac{2}{\sec^2 t}$ so $\frac{dy}{dx} \Big|_{t=\pi/4} = \frac{2}{2} = 1$

6. $r = 2 \sin \theta \Rightarrow r^2 = 2r \sin \theta$

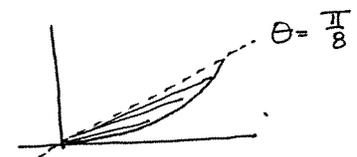
$$x^2 + y^2 = 2y$$

- or - $x^2 + y^2 - 2y + 1 = 1 \leftarrow \text{complete the square}$
 $x^2 + (y-1)^2 = 1$

7.



9. By symmetry we only compute the arc of



$$16 \cdot \frac{1}{2} \int_0^{\pi/8} \sin^2 2\theta d\theta = 4 \int_0^{\pi/8} (1 - \cos 4\theta) d\theta$$

$$= 4 \left(\theta - \frac{1}{4} \sin 4\theta \right) \Big|_0^{\pi/8}$$

$$= \frac{\pi}{2} - 1$$

8. See #7.B on the review

$$\frac{1}{2} (2\sqrt{3} + \ln(2+\sqrt{3}))$$

1.A. $\int x^2 e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \int \frac{1}{2} e^{2x} dx$

$$\left. \begin{array}{l} u = x^2 \quad dv = e^{2x} dx \\ du = 2x dx \quad v = \frac{1}{2} e^{2x} \end{array} \right\} \begin{array}{l} u = x \quad dv = e^{2x} dx \\ du = dx \quad v = \frac{1}{2} e^{2x} \end{array} = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C$$

B. $\int_0^1 \frac{dx}{(1+x^2)^2} = \int_0^{\pi/4} \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int_0^{\pi/4} \cos^2 \theta d\theta = \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/4} = \frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{2} \right)$

let $x = \tan \theta$
 $dx = \sec^2 \theta d\theta$
 $0 \mapsto 0$
 $1 \mapsto \frac{\pi}{4}$

$$= \frac{\pi+2}{8}$$

C. $\int \tan^2 \theta \sec^4 \theta d\theta = \int u^2 (1+u^2) du = \frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} + C$

let $u = \tan \theta$
 $du = \sec^2 \theta d\theta$

D. $\int \frac{2x}{(1+x)(x^2+1)} dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{1}{x+1} \right) dx = \frac{1}{2} \ln(x^2+1) + \arctan x - \ln|x+1| + C$

$$\frac{2x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 2x = A(x^2+1) + (Bx+C)(x+1)$$

let $x = -1 \Rightarrow A = -1$

Now equate coeff.

$$\frac{x^2}{0 = A+B} \Rightarrow B = 1 \quad \frac{x^0}{0 = A+C} \Rightarrow C = 1$$

2. $x = t+1$
 $y = t^2 - 3 \Rightarrow y = (x-1)^2 - 3$

3. $x = 2 \arcsin t$
 $x' = \frac{2}{\sqrt{1-t^2}}$

$y = \ln(1-t^2)$
 $y' = \frac{-2t}{1-t^2}$

$$S = \int_0^{1/2} \frac{2}{1-t^2} dt = \int_0^{1/2} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt = -\ln|1-t| + \ln|1+t| \Big|_0^{1/2} = -\ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right) = \ln 3$$

$$(x')^2 + (y')^2 = \frac{4}{1-t^2} + \frac{4t^2}{(1-t^2)^2} = \frac{4(1-t^2) + 4t^2}{(1-t^2)^2} = \frac{4}{(1-t^2)^2} \quad \Bigg/ \quad \frac{2}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} \Rightarrow 2 = A(1+t) + B(1-t)$$

let $t = 1 \Rightarrow A = 1$
 $t = -1 \Rightarrow B = 1$

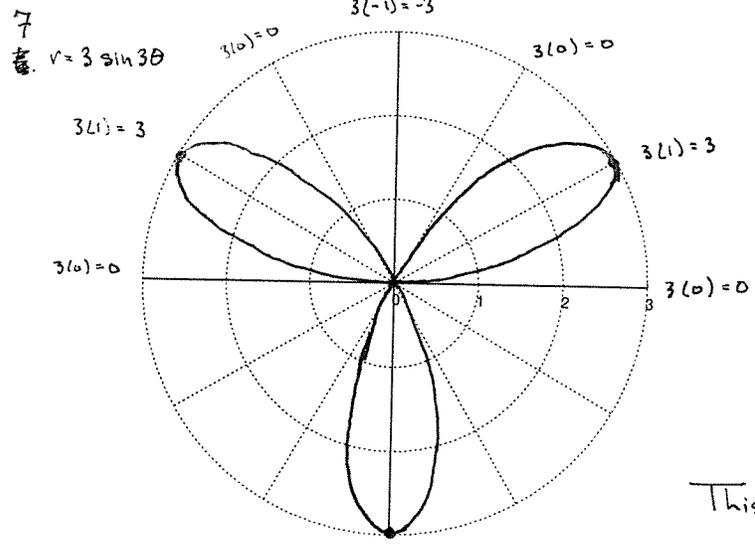
4. $x = 3t$ $y = e^{2t}$
 $x' = 3$ $y' = 2e^{2t}$

Speed = $\sqrt{9 + 4e^{4t}}$ = $\sqrt{9 + 16} = 5$
 $t = \frac{1}{2} \ln 2$ $t = \frac{1}{2} \ln 2$

Note $\ln \sqrt{2} = \frac{1}{2} \ln 2$

5. $\frac{dy}{dx} = \frac{2e^{2t}}{3}$ so $\frac{dy}{dx} = \frac{4}{3}$
 $t = \frac{1}{2} \ln 2$

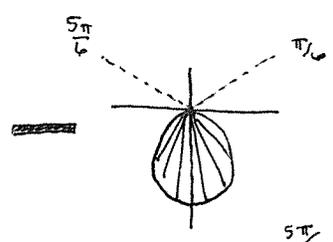
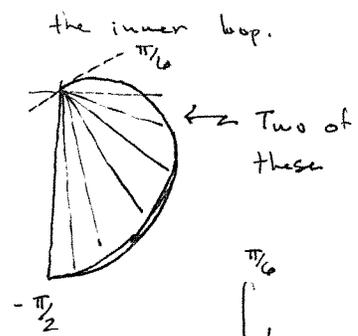
6. $r = 2 \sec \theta \Rightarrow r \cos \theta = 2$
 so $x = 2$



8. $r = e^{-\theta}$
 $r' = -e^{-\theta}$
 $r^2 + (r')^2 = e^{-2\theta} + e^{-2\theta} = 2e^{-2\theta}$
 $ds = e^{-\theta} \sqrt{2} d\theta$
 $s = \int_{-\infty}^0 e^{-\theta} \sqrt{2} d\theta = -\sqrt{2} e^{-\theta} \Big|_{-\infty}^0 = \lim_{R \rightarrow -\infty} (-\sqrt{2} + \sqrt{2} e^{-R}) = \infty$

This was a typo, should have been $r = e^{\theta}$ for $-\infty < \theta \leq 0$ in which case $s = \sqrt{2}$, similar to above. (Typo fixed, depends when you printed your copy.)

9. This is tricky. First we find the area inside the outer loop, then subtract away the area inside the inner loop.



$$= \int_{-\pi/2}^{\pi/6} (1 + 2 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\pi/6}^{5\pi/6} (1 + 2 \sin \theta)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/6} (3 + 4 \sin \theta - 2 \cos 2\theta) d\theta - \frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 + 4 \sin \theta - 2 \cos 2\theta) d\theta$$

$$= (3\theta + 4 \cos \theta - \sin 2\theta) \Big|_{-\pi/2}^{\pi/6} - (3\theta + 4 \cos \theta - \sin 2\theta) \Big|_{\pi/6}^{\pi/2} = \pi + 3\sqrt{3}$$