Math 172 In-Class **Worksheet 10p**

Section: 10.3

It is useful to know that for large n, any $a > 0$ and any $b > 1$, the following are true

$$\ln n < n^a \text{ and } n^a < b^n.$$

1. For the given series and large n, which of the following are useful when using a Direct Comparison? Why?

   (a) $\sum \frac{\ln n}{n^2}$, $\quad 0 < \frac{\ln n}{n^2} < \frac{n}{n^2} = \frac{1}{n}$ Not useful, smaller than divergent is useless.

   (b) $\sum \frac{\ln n}{n^2}$, $\quad 0 < \frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{1.5}}$ Useful, smaller than convergent is useful.

   (c) $\sum \frac{n^{37}}{2^n}$, $\quad 0 < \frac{n^{37}}{2^n} < \frac{2^n}{2^n} = 1$ Not useful, smaller than divergent is useless.

   (d) $\sum \frac{n^{37}}{2^n}$, $\quad 0 < \frac{n^{37}}{2^n} < \frac{(1.5)^n}{2^n} = \left(\frac{1.5}{2}\right)^n$ Useful, smaller than convergent is useful.

   (e) $\sum \frac{1}{n \ln n}$, $\quad 0 < \frac{1}{n^{1.1}} < \frac{1}{n \ln n}$ Not useful, bigger than convergent is useless.

   (f) $\sum \frac{1}{n \ln n}$, $\quad 0 < \frac{1}{n \ln n} < \frac{1}{n}$ Not useful, smaller than divergent is useless

2. For the following series, specify what series you would compare each to (using either the direct or limit comparison test), and then based on your comparison, decide if the series converges or diverges.

   (a) $\sum_{n=0}^{\infty} \left(\frac{1}{2 + \sin^2 n}\right)^n$ compare to $\sum \frac{1}{2^n}$ so it CONVERGES/DIVERGES.

   (b) $\sum_{n=1}^{\infty} \frac{3}{n^2 + \sqrt{n}}$ compare to $\sum \frac{1}{n^2}$ so it CONVERGES/DIVERGES.

   (c) $\sum_{n=1}^{\infty} \frac{3}{n5^n}$ compare to $\sum \frac{1}{5^n}$ so it CONVERGES/DIVERGES.

   (d) $\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^3 + 5n}}$ compare to $\sum \frac{1}{n^{1/2}}$ so it CONVERGES/DIVERGES.

3. Consider the series

$$\sum \frac{n \pm 1}{n^3 \mp 1}.$$

(a) What series should we compare it to, and do we expect it to converge or diverge?

$\frac{n \pm 1}{n^3 \mp 1} \sim \frac{n}{n^3} = \frac{1}{n^2}$, $\sum \frac{1}{n^2}$ converges, so $\sum \frac{n \pm 1}{n^3 \mp 1}$ should as well.

(b) Show $\sum \frac{n-1}{n^3+1}$ converges using the Direct Comparison Test.

For large n, $0 < \frac{n-1}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$.

Since $\sum \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$,

by comparison $\sum \frac{n-1}{n^3+1}$ also converges.

(c) Show $\sum \frac{n+1}{n^3-1}$ converges using the Limit Comparison Test.

For large n, $\frac{n+1}{n^3-1}$ & $\frac{1}{n^2}$ are both positive, so the Limit Comparison Test applies. To that end we consider

$\lim_{n \to \infty} \left(\frac{n+1}{n^3-1} \Big/ \frac{1}{n^2} \right) = 1$, which is positive & finite.

Since $\sum \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$, by the Limit Comparison Test, $\sum \frac{n+1}{n^3-1}$ also converges.