10 Sequences and Series

10.1 Sequences

Dichotomy Paradox, Zeno 490-430 BC: To travel a distance of 1, first one must travel 1/2, then half of what remains, i.e. 1/4, then half of what remains, i.e. 1/8, etc. Since the sequence is infinite, the distance cannot be traveled.

Remark. The steps are terms in the **sequence**.

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$$

Sequences of values of this type is the topic of this first section.

Remark. The sum of the steps forms an infinite series, the topic of Section 10.2 and the rest of Chapter 10.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$$

We will need to be careful, but it turns out that we can indeed walk across a room!

Definition 10.1.1. A **sequence** is a function with domain $\mathbb{N}=\{1,2,3,\dots\}$, the Natural Numbers.

Examples and Notation:

$\{2, 4, 6, 8, \dots\} = \{2n\}_{n=1}^{\infty}$ or often just $a_n = 2n$

$\{-1, 1, -1, 1, \dots\} = \{\cos(\pi n)\}$ $c_n = \cos \pi n$

if no index, assume $n=1$

Fibonacci sequence

$F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

$\{1, 1, 2, 3, 5, 8, \dots\}$

Definition 10.1.2. We say the sequence $\{a_n\}$ converges to L and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$, there exists M such that $|a_n - L| < \epsilon$ when $n > M$. If the limit does not exist, we say the sequence diverges. If $L = \infty$, we say the sequence diverges to infinity.

Examples:

$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \right\} = \left\{ \frac{1}{2^n} \right\}$ converges to 0,

or $\frac{1}{2^n} \longrightarrow 0$ as $n \longrightarrow \infty$

$\{1, 1, 2, 3, 5, \cdots\} = \{F_n\}$ diverges to infinity

or $F_n \longrightarrow \infty$ as $n \longrightarrow \infty$

$\{-1, 1, -1, 1, \cdots\} = \{\cos(\pi n)\}$ diverges since it oscillates

$\left\{ 1 + \left(\frac{-1}{2}\right)^n \right\}_{n=0}^{\infty} = \left\{ 2, \frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{17}{16}, \cdots \right\}$

converges to 1

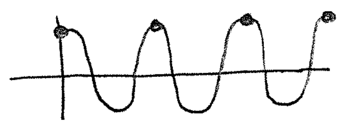
Theorem 10.1.1. *If* $\lim_{x\to\infty} f(x) = L$, *then* $\lim_{n\to\infty} f(n) = L$.

Remark. The implication does not work the other direction, i.e.
$\lim_{n\to\infty} f(n) = L \not\Rightarrow \lim_{x\to\infty} f(x) = L$, for example:

Think of $f(x)$ as a continuous curve

Think of $f(n)$ as a collection of dots

Although $\lim_{n\to\infty} \cos(2\pi n) = 1$, $\lim_{x\to\infty} \cos(2\pi x)$ does not exist



Example 10.1.1. Show $\left\{\frac{\ln n}{n}\right\}$ converges.

Since $\lim_{n\to\infty} \frac{\ln n}{n}$ is of the form "$\frac{\infty}{\infty}$" it is indeterminate.

In order to apply L'H's rule, we consider the continuous case

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Since $\lim_{x\to\infty} \frac{\ln x}{x} = 0$, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

Remark. Do not apply L'Hopital's Rule to terms of a sequence. Sequences are not differentiable functions, not even continuous.

Many previous results regarding limits apply in the sequence case as well. For convenience they are summarized here.

Theorem 10.1.2. *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$ *then*

- $\lim_{n\to\infty} (a_n + b_n) = A + B$,
- $\lim_{n\to\infty} (a_n b_n) = AB$,
- $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ *provided* $B \neq 0$, *and*
- $\lim_{n\to\infty} (ca_n) = cA$ *for any constant* c.

Theorem 10.1.3 (Squeeze Theorem). *If* $a_n \le b_n \le c_n$ *for* $n \ge M$, $\lim_{n\to\infty} a_n = L$, *and* $\lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.

Example 10.1.2. Since $-|a_n| \le a_n \le |a_n|$, by the Squeeze Theorem, if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 10.1.3. Use the Squeeze Theorem to show $\left\{\frac{3^n}{n!}\right\}$ converges.

Initially note that $n!$, n factorial, is defined recursively as

$$0! = 1, \quad n! = n\,(n-1)! \text{ for } n \geq 1$$

Informally, $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

Consider the 5^{th} term in the sequence

$$0 < \frac{3^5}{5!} = \frac{3}{5}\cdot\underbrace{\frac{3}{4}\cdot\frac{3}{3}}_{<1}\cdot\frac{3}{2}\cdot\frac{3}{1} < \frac{3}{5}\cdot\frac{3}{2}\cdot\frac{3}{1}$$

In general the n^{th} term is

$$0 < \frac{3^n}{n!} = \frac{3}{n}\cdot\frac{3}{n-1}\cdots\frac{3}{3}\cdot\frac{3}{2}\cdot\frac{3}{1} < \frac{3}{n}\cdot\frac{9}{2}$$

Since $\frac{3}{n}\cdot\frac{9}{2} \longrightarrow 0$ as $n \longrightarrow \infty$, by the Squeeze Theorem

$$\frac{3^n}{n!} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

One more result from earlier is useful for us.

Theorem 10.1.4. *If f is continuous and $a_n \to L$ as $n \to \infty$, then $f(a_n) \to f(L)$ as $n \to \infty$.*

Example 10.1.4. Find $\lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right)$.

$$= \ln\left(\lim_{n\to\infty} \frac{n}{2n+1}\right)$$

divide numerator & denominator by n

$$= \ln\left(\lim_{n\to\infty} \frac{1}{2+1/n}\right)$$

$$= \ln\left(1/2\right)$$

Definition 10.1.3. Bounded.

- $\{a_n\}$ is **bounded above** if there exist M such that $a_n \leq M$ for all n.
- $\{a_n\}$ is **bounded below** if there exist N such that $a_n \geq N$ for all n.
- $\{a_n\}$ is **bounded** if it is bounded above and below.

Examples:

$\{1, 2, 3, 4, 5, \cdots\}$ is bounded below by 1

$\{-1, -1/2, -1/3, -1/4, \cdots\}$ is bounded,
above by 0 & below by -1

$\{1, 1, 1, 1, \cdots\}$ is bounded
above & below by 1

Definition 10.1.4. Monotone.

- $\{a_n\}$ is **increasing** if $a_n \leq a_{n+1}$ for all n.
- $\{a_n\}$ is **decreasing** if $a_n \geq a_{n+1}$ for all n.
- $\{a_n\}$ is **monotone** if either of the above hold.

Examples:

$\{1, 2, 3, 4, \cdots\}$ is increasing

$\{-n^2\} = \{-1, -4, -9, -16, \cdots\}$

is decreasing and bounded above

$\{-1, 1, -1, 1, -1, 1, \cdots\}$ is not monotone

but is bounded

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\}$ is decreasing and bounded

The two previous definitions are useful by themselves, but combined they give us the one Big Gun of the theory of sequences.

Theorem 10.1.5 (Monotone Convergence Theorem). *If $\{a_n\}$ is bounded and monotone, then it converges.*

We finish our survey of sequences with five limits, many of which will prove useful for the remainder of this chapter.

Theorem 10.1.6. *Let* $p > 0$.

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{p} = 1$
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
4. *If* $|r| < 1$, $\lim_{n\to\infty} r^n = 0$.
5. $\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$

Remark. Parts 2 and 3 imply that if $P(n)$ is any polynomial in n, then $\sqrt[n]{|P(n)|} \to 1$ as $n \to \infty$. This will be particularly useful in an upcoming section.

Selected proofs.

Proof of 3. We consider the continuous version,

$$\lim_{x\to\infty} x^{1/x} = e^{\ln\left(\lim_{x\to\infty} x^{1/x}\right)} = e^{\lim_{x\to\infty}\left(\ln x^{1/x}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{1}{x}\ln x\right)} = e^{\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)}$$

$$\overset{L'H}{=} e^{\lim_{x\to\infty}\left(\frac{1}{x}\right)} = e^0 = 1$$

Proof of 5. We again consider the continuous version,

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = e^{\ln\left(\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x\right)} = e^{\lim_{x\to\infty}\ln\left(1+\frac{c}{x}\right)^x}$$

$$= e^{\lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{c}{x}\right)}{1/x}\right)} \overset{L'H}{=} e^{\lim_{x\to\infty}\left(\frac{\frac{1}{1+c/x}\cdot\frac{-c}{x^2}}{-1/x^2}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{c}{1+c/x}\right)} = e^c$$

10.2 Introduction to Series

We return to the Dichotomy Paradox. It is clear that I can walk across the room, so we should be able to do something like this.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$$

Of course, we will need to formalize this. We start with the standard sigma notation for finite sums

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

For example,

$$\sum_{n=2}^{5} n^2 = \underset{n=2}{\underset{\uparrow}{4}} + 9 + 16 + \underset{n=5}{\underset{\uparrow}{25}}$$

We will define the **Nth partial sum** to be $S_N = \sum_{n=?}^{N} a_n$.

Remark. The starting index is not particularly important. Although many examples will start at $n = 1$ or $n = 0$, there is no reason it cannot start at $n = 47$. Additionally, there is nothing special about the index n. We often use i, j, k, or m as well.

For example, the 5th partial sum for the following series are

$$S_5 = \sum_{k=2}^{5} k = 2 + 3 + 4 + 5$$

$$S_5 = \sum_{k=1}^{5} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$$

$$S_5 = \sum_{k=4}^{5} \frac{1}{k^3-1} = \frac{1}{63} + \frac{1}{124}$$

We would like to define an **infinite series**, i.e.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The standard 172 idea holds, approximate with partial sums and take the limit.

Definition 10.2.1. Let $S_N = \sum_{n=1}^{N} a_n$ be the sequence of Nth partial sums.

- If $\lim_{n\to\infty} S_n = S$, we say $\sum_{n=1}^{\infty} a_n$ **converges** to S and write $\sum_{n=1}^{\infty} a_n = S$.
- If $\lim_{n\to\infty} S_n = \infty$, we say $\sum_{n=1}^{\infty} a_n$ **diverges** to infinity and write $\sum_{n=1}^{\infty} a_n = \infty$.
- If $\lim_{n\to\infty} S_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ **diverges**.

One type of series that we can approach using the definition directly are telescoping series.

Example 10.2.1. Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$.

By partial fractions $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$ so $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$S_1 = 1 - \frac{1}{3}$

$S_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$

$S_3 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$

$S_4 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$

$\vdots$

$S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$ $\qquad S = \lim_{N\to\infty} S_N = 3/2$

so the sum of the series is $3/2$.

There is one additional type of series that we can use the definition directly for, they are the topic of the following section. For now, we turn our attention to one issue of theoretical importance and finally one fundamental example.

Theorem 10.2.1. *If $\sum a_k$ converges, $a_k \to 0$ as $k \to \infty$.*

Remark. Convergence or divergence of a series depends on the behavior of the tail of the series, i.e. we can throw away the first ten, or hundred, or even million terms and not change the convergence of the series. If the index is immaterial to the topic at hand, as in the theorem above, we will often supress the notation for convenience.

Proof.

Assume $\lim_{N \to \infty} S_N = S$, i.e. the series converges.

Now $a_N = S_N - S_{N-1}$, taking the limit as $N \longrightarrow \infty$

we have $\lim_{N \to \infty} a_N = \lim_{N \to \infty} S_N - \lim_{N \to \infty} S_{N-1} = S - S = 0$

The above theorem is typically used in its contrapositive form.

Theorem 10.2.2 (Test for Divergence). *If $\lim_{n \to \infty} a_n \neq 0$, then $\sum a_n$ diverges.*

Remark. We are saying that the terms going to zero is a **necessary** condition for the convergence of a series. However, as the next example shows, it is not a **sufficient** condition.

Example 10.2.2. The **Harmonic Series diverges**, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

Proof.

We consider a subsequence of the partial sums and show it diverges to infinity.

$$S_{2^0} = S_1 = 1$$

$$S_{2^1} = S_2 = 1 + \frac{1}{2}$$

$$S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$

$$\geq 1 + 2\left(\frac{1}{2}\right)$$

$$S_{2^3} = S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2}$$

$$\geq 1 + 3(1/2)$$

$$\vdots$$

$$S_{2^n} \geq 1 + n\left(\frac{1}{2}\right)$$

Since $1 + \frac{n}{2} \longrightarrow \infty$ as $n \longrightarrow \infty$,

the sequence of partial sums diverges and hence the series diverges.