10.6 Ratio and Root Tests

In this section we develop two tests useful for determining the convergence or divergence of series with a particular emphasis on power series. Both are generalizations of the geometric series from section 10.3.

Theorem 10.6.1 (The Ratio Test). *Let* $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

1. *if* $\rho < 1$, $\sum a_n$ *converges.*
2. *if* $\rho > 1$, $\sum a_n$ *diverges.*
3. *if* $\rho = 1$ *or the limit does not exist, the test is inconclusive.*

Sketch of Proof.

For large N $\quad |a_{N+1}| \sim \rho |a_N|$, $\quad |a_{N+2}| \sim \rho^2 |a_N|$, $\ldots$

$$\sum a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N} |a_n| \sim \underbrace{\sum_{n=0}^{N-1} a_n}_{\text{finite}} + \underbrace{\sum_{n=N}^{\infty} \rho^{n-N} |a_N|}_{\text{geometric}}$$

Theorem 10.6.2 (The Root Test). *Let* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

1. *if* $L < 1$, $\sum a_n$ *converges.*
2. *if* $L > 1$, $\sum a_n$ *diverges.*
3. *if* $L = 1$ *or the limit does not exist, the test is inconclusive.*

Sketch of Proof.

As above $\quad |a_N| \sim L^n$

Initial Examples. Show the following series converge.

$$\sum_{n=0}^{\infty} \left(\frac{n+1}{2n+3}\right)^n$$

Root Test

$$\sqrt[n]{|a_n|} = \left(\frac{n+1}{2n+3}\right) \xrightarrow[n\to\infty]{} \frac{1}{2} < 1, \text{ so } \sum \left(\frac{n+1}{2n+3}\right)^n \text{ converges}$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Ratio Test

$$\left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$$

$$\text{so } \sum_{n=0}^{\infty} \frac{2^n}{n!} \text{ converges}$$

Example 10.6.1. In the previous section we showed that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Show that the series converges for all x, i.e. the radius of convergence is infinite.

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0 < 1 \quad \text{for all } x$$

Both tests are useful for determining the radius of convergence of power series. For know power series, there is no need to reinvent the wheel, but to determine the radius of convergence of an unknown series either of the tests is our first step.

There are two important points to note. First, both tests are inconclusive at the endpoints. Neither will give us any insight into the **interval of convergence**. To address that issue we will have to develop more subtle tests; that will be the topic of the next few sections. Second, although the Root Test seems trickier, we have some nice results that streamline the process; see Theorem 10.1.6. In particular...

$$\lim_{n\to\infty} \left(1+\frac{c}{n}\right)^n = e^c \qquad \underline{\text{Note:}}\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} = e^{-1}$$

$$\sqrt[n]{|\text{constant}|} < \sqrt[n]{|\text{logs}|} < \sqrt[n]{|\text{roots}|} < \sqrt[n]{|\text{polynomials}|} < \sqrt[n]{n^K}$$

$\downarrow_{n\to\infty}$ 1 $\qquad\qquad$ $\downarrow_{n\to\infty}$ 1

By Squeeze Thm, everything in the middle does as well.

Power Series Examples. Find the radius of convergence for the following.

$$\sum_{n=1}^{\infty} \frac{n^2(x-2)^{2n}}{9^n}$$

Ratio Test

$$\left| \frac{(n+1)^2 (x-2)^{2n+2}}{9^{n+1}} \cdot \frac{9^n}{n^2 (x-2)^{2n}} \right| = \left(\frac{n+1}{n}\right)^2 |x-2|^2 \cdot \frac{1}{9} \xrightarrow[n\to\infty]{} \frac{|x-2|^2}{9} < 1$$

So $|x-2|^2 < 9$ or $|x-2| < 3$

or $(-1, 5)$

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2} (x+1)^n$$

Root Test

$$\sqrt[n]{|a_n|} = \left(\frac{n}{n+2}\right)^n |x+1| = \frac{|x+1|}{\left(\frac{2+n}{n}\right)^n} = \frac{|x+1|}{\left(1+\frac{2}{n}\right)^n} \xrightarrow[n\to\infty]{} \frac{|x+1|}{e^2} < 1$$

So $|x+1| < e^2$

Two Additional Examples. Find the radius of convergence for the following.

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2+1}$$

Ratio Test

$$\left| \frac{(2x)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(2x)^n} \right| \longrightarrow |2x| < 1$$

$$|x| < 1/2$$

Root

$$\sqrt[n]{|a_n|} = |2x|$$

$$\sum_{n=0}^{\infty} \frac{n!x^n}{n^n}$$

Ratio

$$\left| \frac{(n+1)!\, x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\, x^n} \right| = \frac{(n+1)\, n^n}{(n+1)^{n+1}} \cdot |x| = \left(\frac{n}{n+1} \right)^n |x|$$

$$= \left[\left(1 + \frac{1}{n}\right)^n \right]^{-1} |x| \longrightarrow \frac{1}{e} |x| < 1$$

$$|x| < e$$

Homework

In section 10.5 do #5, 11, 17, 19, 23, 29, 37, 41

Exercise 10.6.2. Find the radius of convergence for the following. Express your solution in the form $|x-c| < R$, where possible.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{\ln n(x+4)^n}{3^n}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-1)^n}{n+2}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2x^n}{(2n)!}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{n!x^n}{n^3+1}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n(x+2)^n}{3^n}$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$