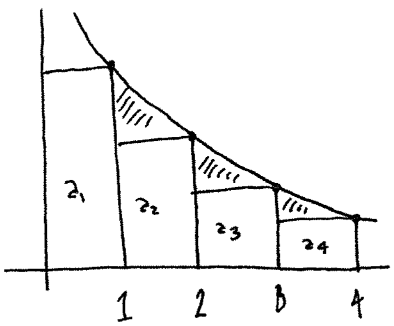
10.7 Convergence of Positive Series

In this section we do a brief survey of methods for testing convergence at endpoints of intervals of convergence.

Theorem 10.7.1 (Integral Test). *Let $a_n = f(n)$, where f is positive and decreasing for $x \geq 1$. $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.*

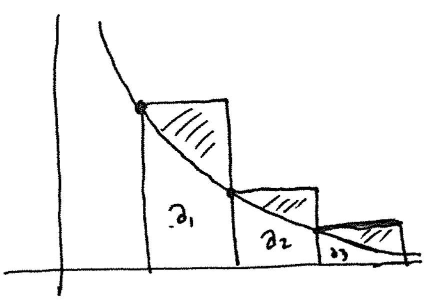
Sketch of Proof.



$$\int_1^\infty f(x)\,dx > a_2 + a_3 + \cdots$$

so $a_1 + \int_1^\infty f(x)\,dx > \sum_{n=1}^\infty a_n$

if $\int_1^\infty f(x)\,dx$ converges so does $\sum_{n=1}^\infty a_n$



$$\sum_{n=1}^\infty a_n > \int_1^\infty f(x)\,dx$$

if $\int_1^\infty f(x)\,dx$ diverges so does $\sum_{n=1}^\infty a_n$

Example 10.7.1. Convergence of p-Series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

For $p \leq 0$, $\frac{1}{n^p} \not\to 0$, so $\sum \frac{1}{n^p}$ diverges by the Divergence Test.

For $p > 0$,

Let $f(x) = \frac{1}{x^p}$. f is positive for $x > 0$, and

$f' = -p x^{-p-1} = \frac{-p}{x^{p+1}} < 0$ for $p > 0$, i.e. f is decreasing,

so the Integral Test applies.

Since $\int_1^{\infty} \frac{1}{x^p} dx$ converges iff $p > 1$,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Theorem 10.7.2 (Direct Comparison Test). *Assume $0 \le a_n \le b_n$ for $n \ge M$.*

1. *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

Examples

Show $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2-1}$ diverges

Since $\displaystyle 0 < \frac{1}{2n} < \frac{n}{2n^2-1}$ & $\displaystyle\sum \frac{1}{2n}$ is a divergent p-series, by comparison $\displaystyle\sum \frac{n}{2n^2-1}$ also diverges.

Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ converges

~~Since~~ $\cancel{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}}$ Since $\displaystyle 0 < \frac{1}{n^2+3n+2} < \frac{1}{n^2}$

& $\displaystyle\sum \frac{1}{n^2}$ is a convergent p-series, by comparison,

$\displaystyle\sum \frac{1}{n^2+3n+2}$ also converges

Theorem 10.7.3 (Limit Comparison Test).

Let $\sum a_n$ & $\sum b_n$ be positive series, i.e. $a_n, b_n \geq 0$.

Assume $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists

1) If $0 < L < \infty$, then $\sum a_n$ converges if & only if $\sum b_n$ converges

2) If $L = 0$, ~~then~~ $\sum a_n$ converges if $\sum b_n$ converges

3) If $L = \infty$, then $\sum b_n$ ~~diverges~~ converge if $\sum a_n$ ~~diverges~~ converges

Examples

Show $\sum \sin\left(\frac{1}{n}\right)$ diverges.

Since $\sum \sin\left(\frac{1}{n}\right)$ & $\sum \frac{1}{n}$ are both positive series the Limit Comparison Test applies. To that end, we compute the limit

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{LH}{=} \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1.$$

Since the limit is positive & finite and since $\sum \frac{1}{n}$ diverges, by the L.C.T. $\sum \sin\left(\frac{1}{n}\right)$ also diverges.

Show $\sum \frac{n+1}{n^3-2}$ converges.