10.3 Geometric Series

An important, perhaps the most important, type of series is a geometric series. We have already seen one example, our walk across the room.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Before we dive into the general theory, we should look closely at this example. As before, we consider the Nth partial sum S_N.

$$S_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N}$$

$$\frac{1}{2} S_N = \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N} + \frac{1}{2^{N+1}}$$

Subtracting gives

$$S_N - \frac{1}{2} S_N = \frac{1}{2} - \frac{1}{2^{N+1}}$$

$$S_N = \frac{\frac{1}{2} - \frac{1}{2^{N+1}}}{\frac{1}{2}} = 1 - \frac{1}{2^N}$$

Taking limits we have

$$S_N = 1 - \frac{1}{2^N} \longrightarrow 1 \quad \text{as} \quad N \longrightarrow \infty$$

i.e.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Theorem 10.3.1 (Geometric Series). *For $c \neq 0$, $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$ for $|r| < 1$ and diverges otherwise.*

Proof. For $r \neq 1$,

$$S_N = c + cr + cr^2 + cr^3 + \cdots + cr^N$$

$$rS_N = cr + cr^2 + cr^3 + \cdots + cr^N + cr^{N+1}$$

Subtracting yields

$$S_N - rS_N = c - cr^{N+1}$$

so $$S_N = \frac{c(1-r^{N+1})}{1-r} \quad \text{for } r \neq 1$$

For $|r| < 1$, $r^{N+1} \longrightarrow 0$ as $N \longrightarrow \infty$ and the result follows

For $|r| > 1$, $|r|^{N+1} \longrightarrow \infty$ as $N \longrightarrow \infty$ and the series diverges.

For $r = 1$ the series is $\sum_{n=0}^{\infty} c = c + c + c + \cdots$ which diverges for $c \neq 0$

For $r = -1$ the series oscillates $c - c + c - c + c - c \cdots$ & diverges for $c \neq 0$.

Remark. The more general result is

$$\sum cr^n = \frac{\text{first term}}{1-r}$$

for $|r| < 1$.

Examples:

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = 3 + \frac{3}{2} + \frac{3}{4} + \cdots = \frac{3}{1 - 1/2} = \frac{3}{1/2} = 6$$

$$\sum_{n=3}^{\infty} \frac{3}{2^n} = \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots = \frac{3/8}{1 - 1/2} = \frac{3}{4}$$

$$\frac{1}{9} + \frac{8}{9^2} + \frac{64}{9^3} + \cdots = \frac{1/9}{1 - 8/9} = 1$$

$$\frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots = \text{diverges since } r = \frac{3}{2} > 1$$

$$\sum_{n=0}^{\infty} \frac{1 + (-2)^n}{3^{2n}}$$

We will need additional machinery to deal with this last example.

Since finite sums and limits are both linear, so are series.

Theorem 10.3.2 (Linearity of Series). *Assume the following series are convergent, then*

- $\sum ca_n = c\sum a_n,$ *and*
- $\sum (a_n + b_n) = \sum a_n + \sum b_n.$

We can now return to the example from the previous page and a similar example.

$$\sum_{n=0}^{\infty} \frac{1+(-2)^n}{3^{2n}} = \sum_{n=0}^{\infty} \frac{1}{9^n} + \sum_{n=0}^{\infty} \left(\frac{-2}{9}\right)^n$$

$$= \frac{1}{1-\frac{1}{9}} + \frac{1}{1-\left(\frac{-2}{9}\right)} = \frac{9}{8} + \frac{9}{11} = \frac{171}{88}$$

$$\sum_{n=0}^{\infty} \frac{3^n+(-4)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{-4}{5}\right)^n$$

$$= \frac{1}{1-\frac{3}{5}} + \frac{1}{1-\left(\frac{-4}{5}\right)} = \frac{5}{2} + \frac{5}{9} = \frac{55}{18}$$

Remark. The assumption that all the series converged in the theorem is necessary. For example,

$$0 = \sum 0 = \sum (1-1) \neq \sum 1 - \sum 1$$

Since neither of the last series converge.

In our discussion of geometric series, the common ratio r was constant. What happens if we let r vary?

Example 10.3.1. Find the values of x for which the following series converges and find what it converges to.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

for $|x| < 1$

Example 10.3.2. Find the values of x for which the following series converges and find what it converges to.

$$\sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n}}{4^n} = \frac{2}{1 - \left(\frac{-x^2}{4}\right)} = \frac{8}{4 + x^2}$$

for $\left|\frac{(-1)^n x^2}{4}\right| < 1$

$$x^2 < 4$$

$$|x| < 2$$

Remark. The two series on this page are representations of functions. They are examples of series we will refer to to as power series, the topic section 10.5.

Homework From section 10.2 in the text, # 23, 25, 27, 29, 33, 39, 43, 47, 49, 57

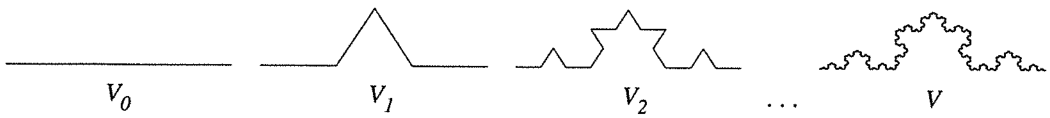


Figure 10.2: von Koch curve

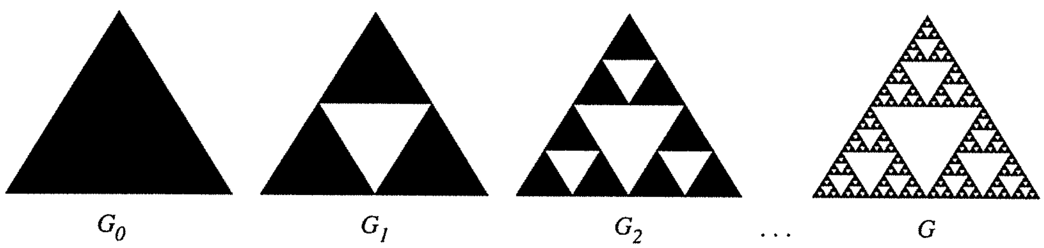


Figure 10.3: Sierpiński gasket

can view the construction as replacing each interval with four intervals of one third the length, see Figure 10.2.

We will soon see that the Cantor set has zero length. Curiously, the snowflake curve has infinite length. Nonetheless, the curve has zero area in the plane. It will be useful to find a measure of the size of this curve, and similar objects, that is a more useful measurement than infinite length or zero area. One such measure of size is the dimension, which we will discuss soon. However, two additional examples are worth familiarizing ourselves with before we dive in.

The final two examples are similar to the construction of the Cantor set and both due to the Polish mathematician Wacław Sierpiński. The Sierpiński gasket is constructed from an equilateral triangle. At each iteration we divide the triangle, or triangles, into four congruent subtriangles and remove the central subtriangle, see Figure 10.3. A Sierpiński carpet is constructed from a square. At each iteration we divide the square, or squares, into nine congruent subsquares and remove the central subsquare, see Figure 10.4.

We will soon see that the area of the gasket and the carpet are both zero. Again we see that area is a poor measure of these objects. A more useful analytic tool

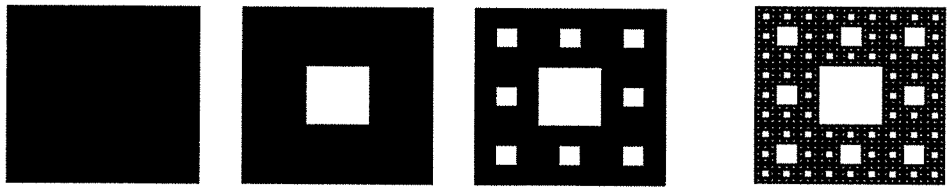


Figure 10.4: Sierpiński carpet

for these types of fractals is their dimension.

Before we consider the concept of dimension, we should verify the claims made earlier.

Exercise 10.4.1. Show the length of the Cantor set is 0.

This is homework.

Exercise 10.4.2. Show the lenght of the von Koch curve is infinite.

Solution. It is clear the line segment V_0 has length 1. At the next iteration we replace the segment by four pieces one third the length, so the length of V_1 is $(4/3)$. The length grows by this factor at each step, so V_2 has length $(4/3)^2$, V_3 has length $(4/3)^3$, and in general V_n has length $(4/3)^n$. Since $(4/3)^n \to \infty$ as $n \to \infty$ we see the length of the von Koch curve V is infinite. This is an example of a divergent geometric sequence, see Theorem 10.1.6.

Exercise 10.4.3. Show the area of the Sierpiński gasket is 0.

This is homework.

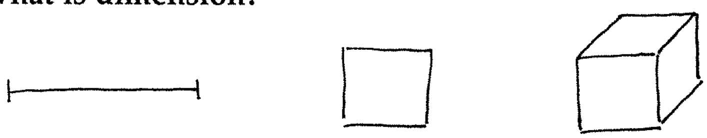
Exercise 10.4.4. Show the area of the Sierpiński carpet is 0.

Solution. The area can be computed by subtracting the removed squares from the total area. Conviently, the area of removed squares form a geometric series.

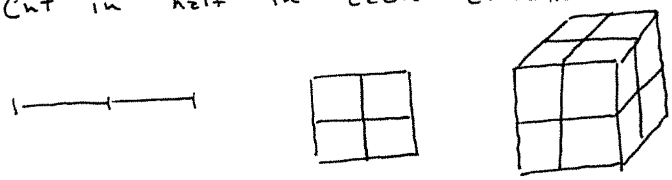
$$\text{Area} = 1 - \left(\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \cdots \right) = 1 - \frac{1/9}{1 - 8/9} = 1 - 1 = 0$$

For these types of mathematical toys, and many real world objects that have self-simliar structures at various scales, a useful measure is the dimension. There are many concepts of dimension, we will discuss a very basic version.

What is dimension?



Cut in half in each direction



How do the number of pieces relate to the scale $\left(\frac{1}{2}\right)$

$$2 = \left(\tfrac{1}{2}\right)^{-1} \qquad 4 = \left(\tfrac{1}{2}\right)^{-2} \qquad 8 = \left(\tfrac{1}{2}\right)^{-3}$$

The exponent of growth is the dimension.

~~so~~ expressed slightly differently, $\frac{1}{1/2} = 2$ so

$$2 = 2^{1} \qquad 4 = 2^{2} \qquad 8 = 2^{3}$$

Roughly,

$$\text{number of pieces} = (1/(\text{`size' of pieces}))^{\text{dimension}}$$

Solving for the dimension in the previous gives.

$$\dim X = \frac{\ln(\text{number of pieces})}{-\ln(\text{'size' of pieces})} \qquad (*)$$

Or formally, if X is self similar shape made of N copies of itself, each scaled by a similarity with contraction factor r then we define the **similarity dimension** as

$$\dim_S X = \frac{\ln N}{-\ln r}$$

The examples we have seen have the following similarity dimensions.

- Cantor set K, $\dim_S K = \frac{\ln 2}{\ln 3} \approx 0.63$
- von Koch curve V, $\dim_S V = \frac{\ln 4}{\ln 3} \approx 1.26$
- Sierpiński gasket$G$, $\dim_S G = \frac{\ln 3}{\ln 2} \approx 1.58$
- Sierpiński carpet S, $\dim_S S = \frac{\ln 8}{\ln 3} \approx 1.89$

Although similarity dimension is very easy to compute for our examples, it is not very flexible. For more complicated objects, mathematical or real world, other more rigorous methods are needed. But, we are already beyond the scope of the class, so we should return to the topic at hand.

<u>Remark</u> a less rigid version of $(*)$

is $\dim X = \lim_{\text{'size'} \to 0} \frac{\ln(\text{number of pieces})}{-\ln(\text{'size' of pieces})}$