10.5 Introduction to Power Series

An important application of series is that of a power series, i.e. a way to represent a function by a series. In essence, turning a function like e^x into something computable. Although it is useful to think of e^x as the function that is its own derivative and that has slope 1 when $x = 0$, that is hard to use from a computational point of view.

We saw two examples of power series at the end of section 10.3. The most important being

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1. \tag{10.1}$$

The goal of this chapter is to be able to find series representations for functions. In essence, finding 'infinite polynomial' representations of functions.

Definition 10.5.1. A **power series** with center c is a series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

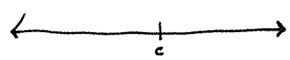
In the power series (10.1) above, $a_n = 1$ for all n and the center is $c = 0$. One thing to note is that the series only converges for $|x| < 1$. This type of restriction on x is typical.

Theorem 10.5.1 (Radius of Convergence). *Every power series*

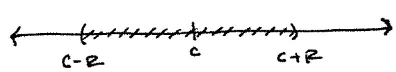
$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

has a **radius of convergence** *R with $R = 0, R > 0$, or $R = \infty$.*

- *If $R = 0$, the series only converges at $x = c$.*



- *If $R > 0$, the series converges for $|x - c| < R$.*



- *If $R = \infty$, the series converges for all $x \in \mathbb{R}$.*

We will be interested in determining the radius of convergence for power series soon, but we start with generating some power series based off of (10.1).

Question: If $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x=0$ does it converge at $x = 5$?

Games we can play with power series.

1. **Multiplication.** Find a power series representation for

$$f(x) = \frac{2x}{1-x}.$$

$$= 2x\left(\frac{1}{1-x}\right) = 2x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 2x^{n+1}$$

for $|x|<1$

2. **Substitution.** Find a power series representation for

$$g(x) = \frac{1}{1+x^2}.$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|-x^2|<1$

i.e. $|x|<1$

3. **Shifting the center.** Find a power series representation centered at $x = 2$ for
$$h(x) = \frac{1}{1-x}.$$

↳ we need $(x-2)$ in the series

$$\frac{1}{1-x} = \frac{1}{1-(x-2)-2} = \frac{1}{-1-(x-2)} = \frac{-1}{1-\left(\frac{x-2}{-1}\right)}$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{x-2}{-1}\right)^n = \sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$$

for $\left|\frac{x-2}{-1}\right| < 1$ i.e. $|x-2| < 1$

One reason power series are so useful is that they are very easy to differentiate and integrate.

Theorem 10.5.2. *Assume*

$$F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

has radius of convergence R. Then F is differentiable and integrable on $(c-R, c+R)$ with

$$F'(x) = \sum_{n=1}^{\infty} n a_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

and, for any constant A

$$\int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

Furthermore, all three series have the same radius of convergence.

Example 10.5.1. Find a power series representation for $\ln(1+x)$.

First, $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

Since

$$\frac{d}{dx}\left(\ln(1+x)\right) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = A + \int \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

since $\ln(1) = 0$, $A = 0$ & we have

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Exercise 10.5.2. Find a power series representation for $\frac{1}{(1+x)^2}$.

* Hint: $\frac{d}{dx}\left[\frac{1}{1+x}\right] = \frac{-1}{(1+x)^2}$

Example 10.5.3. Find a power series representation for $\arctan x$.

We saw earlier $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|x| < 1$

So $\arctan x = A + \int \frac{1}{1+x^2}\,dx$

$$= A + \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Since $\arctan 0 = 0$, $A = 0$

So $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| < 1$

Remark. Although the radius of convergence does not change when we integrate, it is possible that the **interval of convergence** does. We will discuss the convergence at endpoints soon, but for now we simply note that the representation of $\arctan x$ is valid for $x \in [-1, 1]$ which gives the following curious result.

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Example 10.5.4. Find a power series representation for e^x.

We start by a quick diversion into differential equations, in particular it is major result in the subject that there is a unique solution to an Inital Value Problem (IVP)of the following type

$$y' = y, \quad y(0) = 1$$

i.e. there is a unique function that is its own derivative with slope one when $x = 0$ The function e^x clearly satifies the IVP and so is the unique solution.

We will put that knowledge on hold for a minute and search for a power series solution to the IVP. Using power series to solve differential equations is a fundamental tool in the subject.

Assume there is a series solution

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

then $$F'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Substituting into the IVP & equating coefficients gives

$$a_0 = a_1, \quad a_1 = 2a_2, \quad a_2 = 3a_3, \quad \cdots$$

or $\cancel{a_0 =} \frac{a_1}{1}$ in general $a_{n-1} = n a_n$

or $$a_n = \frac{a_{n-1}}{n}$$

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a_0 is arbitrary, but the remaining terms can be determined recursively from it.

$$a_1 = a_0$$

$$a_2 = \frac{a_1}{2} = \frac{1}{2} a_0$$

$$a_3 = \frac{a_2}{3} = \frac{1}{3!} a_0$$

$$\vdots$$

$$a_n = \frac{1}{n!} a_0$$

We can choose a_0 arbitrarily, and use (10.2) to find the other coefficients in terms of a_0.

$$a_0$$

$$a_1 = \frac{a_0}{1} = a_0$$

$$a_2 = \frac{a_1}{2} = \frac{1}{2} a_0$$

$$a_3 = \frac{a_2}{3} = \frac{1}{3\cdot 2} a_0 = \frac{1}{3!} a_0$$

$$a_4 = \frac{a_3}{4} = \frac{1}{4}\cdot\frac{1}{3!} a_0 = \frac{1}{4!} a_0$$

$$\vdots$$

$$a_n = \frac{1}{n!} a_0$$

$$\text{so} \quad f(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n$$

We conclude that

$$F(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We also want it to satisfy the inital data, i.e. $F(0) = 1$. Since $F(0) = a_0$ we conclude that $a_0 = 1$ and

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We are left with one major concern; for what values of x does this series converge? That question will be the topic of the following section.

Homework.

Exercise 10.5.4. True / False

1. **T / F :** If $\sum a_n(x-3)^n$ converges for $x=5$ it converges for $x=0$.
2. **T / F :** If $\sum a_n(x+3)^n$ converges for $x=5$ it converges for $x=0$.
3. **T / F :** There exists a power series that only converges for $x>0$.

Exercise 10.5.5. If a power series converges for $-4<x<2$, what is the center and radius of convergence? $C=-1$, $R=3$

Exercise 10.5.6. Find a power series representation for the following functions. Specify where each series converges. Unless otherwise specified, center the series at $x=0$.

1. $f(x)=\dfrac{1}{1+4x} = \sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-4)^n x^n$ for $|x|<\frac{1}{4}$
2. $f(x)=\dfrac{1}{8+x^3} = \sum$
3. $f(x)=\dfrac{3x^2}{2+x^4}$
4. $f(x)=\dfrac{1}{1-x}$ centered at $x=4$
5. $f(x)=\dfrac{1}{(1+x)^2}$ Hint: $\frac{d}{dx}\left(\frac{1}{1+x}\right)=\frac{-1}{(1+x)^2}$

Exercise 10.5.7. Integrate $\int e^{x^2}\,dx$ using a series.

$$e^{x^2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

$$\int e^{x^2}\,dx = A+\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)\,n!}$$