Show $\sum \frac{n+1}{2n^3-2}$ converges.

Since $\sum \frac{n+1}{2n^3+2}$ & $\sum \frac{1}{n^2}$ are both positive series, the Limit Comparison Test applies. To that end we compute the limit

$$\lim_{n\to\infty} \frac{n+1}{2n^3+2} \Big/ \frac{1}{n^2} = \frac{1}{2}.$$

Since the limit is positive & finite, both series have the same convergence behavior. We know $\sum \frac{1}{n^2}$ is a convergent p-series, so by the L.C.T.

$\sum \frac{n+1}{2n^3-2}$ also converges

Example 10.7.2. Using the Ratio Test we can show that $\sum_{n=1}^{\infty} \frac{(x-2)^n}{2^n n}$ converges for $|x-2| < 2$ and diverges for $|x-2| > 2$. Does it converge when $|x-2| = 2$?

When $x = 4$ we have

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{which diverges}$$

When $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Our current tools do not address this situation, but the following section will.

Exercise 10.7.3. From section 10.3 in the text do # 79 using the Integral Test, 19, 21, 25, 31, 39, 41, 43, 47.

10.8 Conditional and Absolute Convergence

Initially we note that everything we discussed regarding positive series $\sum a_n$, i.e. $a_n > 0$, in the previous section applies to negative series $\sum b_n$, i.e. $b_n < 0$, after we factor out the negative (series are linear). For series with both signs, we will need additional tools.

Definition 10.8.1. $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

Remark. A convergent positive series is absolutely convergent by definition.

Theorem 10.8.1. *If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.*

Proof. $-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$

$\sum |a_n|$ converges $\Rightarrow$ $\sum 2|a_n|$ converges $\Rightarrow$ $\sum (a_n + |a_n|)$ converges by comparison

so

$\sum a_n = \sum (|a_n| + a_n - |a_n|) = \sum (a_n + |a_n|) - \sum |a_n|$ also converges. ∎

Examples.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$$ converges absolutely since $\sum \frac{1}{n^2}$ converges

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$$ converges absolutely by comparison since $0 < \left|\frac{\cos n}{n^2+1}\right| < \frac{1}{n^2}$

& $\sum \frac{1}{n^2}$ converges

Remark. To use either of the comparison tests we must have nonnegative terms, so often testing for absolute convergence is necessary.

Example 10.8.1.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \cdots$$

is the **Alternating Harmonic Series**. Since the Harmonic Series diverges, the Alternating Harmonic Series does not converge absolutely. But, does it converge? This was the question we left off with in Example 10.7.2.

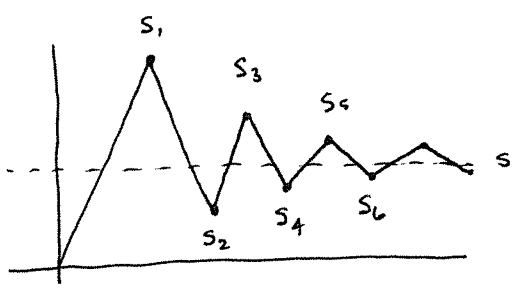
Definition 10.8.2. $\sum a_n$ **converges conditionally** if it converges, but not absolutely.

Definition 10.8.3. For $a_n > 0$, a series of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ is an **alternating series**.

Theorem 10.8.2 (Alternating Series Test). *If $a_n > 0$, a_n decreasing, and $a_n \to 0$ as $n \to \infty$, then $S = \sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges. Additionally, we have the following estimates for $N \geq 1$:*

1. $0 < S < a_1$,
2. $S_{2N} < S < S_{2N+1}$, *and*
3. $|S - S_N| < a_{N+1}$.

Sketch of Proof.



To finish Example 10.8.1 above,

Since $\frac{1}{n} > 0$, $\frac{1}{n} > \frac{1}{n+1}$ & $\frac{1}{n} \xrightarrow[n\to\infty]{} 0$,

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges & hence converges conditionally.

Example 10.8.2. Find the interval of convergence for the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x+2)^n}{(n+1)3^n}. = \frac{1}{1} - \frac{(x+2)}{2\cdot 3} + \frac{(x+2)}{3\cdot 3^2} -$$

As before, we use either the Ratio or Root test to find the radius of convergence.

We use the Root test

$$\sqrt[n]{|a_n|} = \frac{|x+2|}{3}\sqrt[n]{n+1} \xrightarrow[n\to\infty]{} \frac{|x+2|}{3}$$

$$\frac{|x+2|}{3} < 1 \quad \text{when} \quad |x+2| < 3$$

Now we check the endpoints individually.

For $x = -5$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad \text{which diverges by comparison}$$

$$\text{since} \quad 0 < \frac{1}{2n} < \frac{1}{n+1} \quad \& \quad \sum \frac{1}{2n} \text{ diverges.}$$

For $x = 1$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \quad \text{which converges by the AST}$$

$$\text{Since} \quad \frac{1}{n+1} > 0, \quad \frac{1}{n+1} > \frac{1}{n+2} \quad \& \quad \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0.$$

* so $(-5, 1]$ is the interval of convergence.

If we approximate $f(-1)$ with $1 - \frac{1}{6} + \frac{1}{27}$ how far off are we? $|S - S_2| < a_3$

i.e. $\frac{1}{4\cdot 3^3} \approx 0.01$

is it an over estimate or an under? ~~A~~. over

Example 10.8.3. In Example 10.5.2 we found a power series representation for $\arctan x$ by integrating the power series for $1/(1+x^2)$, in particular

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{10.4}$$

which was valid for $|x| < 1$. Theorem 10.5.2 assured us that the radius of convergence remained 1, but it does not tell us anything about the convergence or divergence at the endpoints of the interval. We test the two endpoints now.

At $x = 1$ we have

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which converges by the A.S.T. since

$$\frac{1}{2n+1} > 0, \quad \frac{1}{2n+1} > \frac{1}{2n+3} \quad \& \quad \frac{1}{2n+1} \xrightarrow[n\to\infty]{} 0$$

At $x = -1$ we have

$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ which converges by the A.S.T., see above.

We conclude that (10.4) is valid for $-1 \le x \le 1$.

Example 10.8.4. In Example 10.5.1 we found a power series representation for $\ln(1+x)$ by integrating the power series for $1/(1+x)$, in particular

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{10.5}$$

which was valid for $|x| < 1$. As in the previous example, we would like to check the endpoints for convergence.

At $x = 1$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$ which converges, see example 10.8.2.

At $x = -1$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)}{n+1}$$ which diverges,

see example 10.8.2.

We conclude that (10.5) is valid for $-1 < x \leq 1$. It would be a problem if the series converged at $x = -1$ since we would be computing the value of $\ln 0$.

Example 10.8.5. Show

fix this

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}(-1)^n \tag{10.6}$$

converges conditionally. This argument requires two parts.

First, we must show (10.6) does not converge absolutely.

Although we could use the L.C.T., let's be clever instead

$$0 < \frac{1}{2n} = \frac{n}{n^2+n^2} < \frac{n+1}{n^2+1}.$$ Since $\sum \frac{1}{2n}$ diverges,

by comparison $\sum \frac{n+1}{n^2+1}$ also diverges & hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^2+1}$$ does not converge absolutely.

Second, we must show (10.6) does converge.

To show convergence we use the A.S.T. with $a_n = \frac{n+1}{n^2+1}$.

It is clear $a_n = \frac{n+1}{n^2+1} > 0$,

It is also clear $a_n \longrightarrow 0$ as $n \longrightarrow \infty$.

What is not clear is that the terms decrease. We consider the continuous analog, $f(x) = \frac{x+1}{x^2+1}$. $f'(x) = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2}$

$$= \frac{-x^2-2x+1}{(x^2+1)^2} < 0 \text{ for } x \geq 1.$$

Since $f(n)$ is decreasing, so is a_n.

With that, the A.S.T shows (10.6) converges.

Conditional Convergence is Curious, or, a brief bit a mathematical weirdness.

In the previous example we saw that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

We consider rearranging the terms as follows

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$$

$$= \frac{1}{2} \ln 2$$

This turns out to be the case with all conditionally convergent series, if we rearrange the terms we can change the sum, or even make the series diverge.

Theorem 10.8.3 (Riemann Rearrangement Theorem). *If $\sum a_n$ is a conditionally convergent series, then for any $M \in \mathbb{R}$ there exists a rearrangement of $\{a_n\}$ into $\{b_n\}$, i.e. a one-to-one onto mapping, such that $\sum b_n = M$. Furthermore, there is a different rearrangement of $\{a_n\}$ into $\{c_n\}$ such that $\sum c_n = \infty$.*

Theorem 10.8.4. *If $\sum a_n$ is absolutely convergent, then any rearrangement also converges absolutely and to the same value.*

Remark. Now that we have the appropriate language it should be noted that the conclusion in both the Root and Ratio test is that the series in question converges absolutely when the appropriate limit is less than one. Similarly, power series converge absolutely on their radius of convergence. Since absolute convergence implies convergence, the results as written are true, but not as strong as they are properly.

Homework

Exercise 10.8.6. True or False.

1. **T / F :** If $\sum a_n$ converges conditionally, then $\sum |a_n|$ converges.
2. **T / F :** If $\sum a_n$ converges absolutely, then $\sum |a_n|$ converges.
3. **T / F :** If $\sum |a_n|$ converges, then $\sum a_n$ converges.
4. **T / F :** If $a_n \to 0$ as $n \to \infty$, then $\sum a_n$ converges.
5. **T / F :** If $|a_n| \to 0$ as $n \to \infty$, then $\sum a_n$ converges.

Exercise 10.8.7. Determine if the following CONVERGE ABSOLUTELY, CONVERGE CONDITIONALLY, or DIVERGE. Proper justification will be expected on quizzes and exams, now is a good time to practice.

1. $\sum \dfrac{\sin n \cos 5n}{n^2+1}$
2. $\sum \dfrac{(-1)^n}{\sqrt{n^2+1}}$

Exercise 10.8.8. Find an error estimate for $\left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5}\right)\right| <$ $\frac{1}{7}$.
Hint: See Theorem 10.8.2 and Example 10.8.3.

Exercise 10.8.9. From section 10.6 in the text do # 11, 13, 19, 23 (Note: for $n \geq 1$, $\ln n < n$), 29