10.9 Intuition

In this section we concentrate on trying to build and intuition and understanding of the behavior of series. Although we should be able to craft proper arguments to verify all of the claims we will make, it is important to be able to identify a convergent series. Of equal importance, being able to identify the correct method of showing convergence or divergence.

Methods we have discussed.

1. Test for Divergence.
   Examples.

$\sum n$ diverges since $n \xrightarrow[n\to\infty]{} \infty$

$\sum \frac{1}{n}$ diverges, but the Test for Divergence does not apply.

2. Geometric Series.
   Examples.

$\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = \frac{3}{1-\frac{1}{2}} = 6$ since $|\frac{1}{2}| < 1$,

i.e. it converges

$\sum \frac{2^{2n}}{3^n} = \sum \left(\frac{4}{3}\right)^n$ diverges since $|\frac{4}{3}| > 1$

3. Ratio Test. $\longleftarrow$ useful for factorials!

   Examples.

$$\sum \frac{n+1}{n!}$$

$$\left| \frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1} \right| = \frac{n+2}{n+1} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} \xrightarrow[n\to\infty]{} 0 < 1,$$

so $\sum \frac{n+1}{n!}$ converges by the Ratio Test

4. Root Test. $\longleftarrow$ useful for exponentials & powers

   Examples.

$$\sum \frac{2^n n}{3^n} \qquad \sqrt[n]{|a_n|} = \frac{2\sqrt[n]{n}}{3} \xrightarrow[n\to\infty]{} \frac{2}{3} < 1$$

so $\sum \frac{2^n n}{3^n}$ converges by the ~~Ratio~~ Root Test

$$\sum \left(\frac{n+1}{3n+2}\right)^n \qquad \sqrt[n]{|a_n|} = \frac{n+1}{3n+2} \xrightarrow[n\to\infty]{} \frac{1}{3} < 1$$

so $\sum \left(\frac{n+1}{3n+2}\right)^n$ converges by the Root Test

Question: For what types of series are the Root and Ratio Test inconclusive?

Both fail unless the series has a term at least as big as a exponential.

5. Integral Test.
   Examples.

$$\sum \frac{1}{n^p} \quad \text{converges iff } p>1$$

$$\sum \frac{1}{n(\ln n)^p} \quad \text{converges iff } p>1,$$

this was assigned homework #79 in section §10.3

6. Comparison Tests. ← used for rational functions
   Examples.

$$\sum \frac{n}{n^2+1} \quad \text{diverges} \sim \sum \frac{1}{n}$$

$$\sum \frac{n}{n^3+3} \quad \text{Converges} \sim \sum \frac{1}{n^2}$$

Exercise 10.9.1. For each of the following series determine if they converge or diverge and then choose a test that can be used to show that.

1. $\sum\left(1+\frac{1}{n}\right)^{-n^2}$ Converges / Diverges by the Root Test / Divergence Test. $L = \frac{1}{e}$
2. $\sum\left(1+\frac{1}{n}\right)^{n}$ Converges / Diverges by the Root Test / Divergence Test.
3. $\sum\frac{1}{n\ln n}$ Converges / Diverges by the Ratio Test / Integral Test.
4. $\sum\frac{n+1}{n^2+2}$ Converges / Diverges by the LCT / Ratio Test.
5. $\sum\frac{(-1)^n}{n}$ Converges / Diverges by the Integral Test / AST.
6. $\sum(-n)^n$ Converges / Diverges by the Divergence Test / AST.

Exercise 10.9.2. For the following series, specify what series you would compare each to (either direct or limit comparison) and based on your comparison, decide if it converges or diverges. No formal justification is needed.

1. $\sum_{n=2}^{\infty}\frac{\sqrt{n^2+1}}{n^3+4n}$ compare to $\sum \frac{1}{n^2}$ so it CONVERGES / DIVERGES
2. $\sum_{n=2}^{\infty}\frac{3}{2^n\sqrt{n}}$ compare to $\sum \left(\frac{1}{2}\right)^n$ so it CONVERGES / DIVERGES
3. $\sum_{n=2}^{\infty}\frac{1}{n^2+\sqrt{n}}$ compare to $\sum \frac{1}{n^2}$ so it CONVERGES / DIVERGES
4. $\sum_{n=2}^{\infty}\frac{\sqrt{n^3+3}}{n^2+n}$ compare to $\sum \frac{1}{\sqrt{n}}$ so it CONVERGES / DIVERGES
5. $\sum_{n=2}^{\infty}\frac{2^n}{n3^n}$ compare to $\sum \left(\frac{2}{3}\right)^n$ so it CONVERGES / DIVERGES

Exercise 10.9.3. Choose all series below that...

1. can be shown to **converge** using the Divergence Test. NONE

   (a) $\sum \frac{4 \ln n}{n^2+1}$ (b) $\sum \frac{2}{n}$ (c) $\sum \frac{1}{e^n + n^2}$

2. can be shown to **diverge** using the Divergence Test.

   (a) $\sum \frac{1}{n}$ (b) $\sum \frac{n-1}{n}$ (c) $\sum \frac{1}{\ln n}$

3. can be shown to **converge** using either Comparison Test. NONE

   (a) $\sum \frac{1}{n \ln n}$ (b) $\sum \frac{(-1)^n}{n^2+1}$ (c) $\sum \frac{\ln n}{n}$

4. can be shown to **converge** using the Alternating Series Test.

   (a) $\sum \frac{(-1)^n}{\sqrt{n}}$ (b) $\sum \frac{\cos(\pi n)}{n}$ (c) $\sum \frac{(-1)^n n}{n+1}$

5. can be shown to **diverge** using the Alternating Series Test. NONE

   (a) $\sum \frac{\cos n}{n}$ (b) $\sum \frac{(-1)^n}{n}$ (c) $\sum \frac{(-1)^n n}{n+1}$

6. can be shown to **converge** using the Ratio or Root Test.

   (a) $\sum \frac{n}{n^3+5n}$ (b) $\sum \frac{3^n}{2^{2n+1}+1}$ (c) $\sum \frac{2^n}{n!}$

7. can be shown to **diverge** using the Ratio or Root Test.

   (a) $\sum \frac{n^2}{n^3+5n}$ (b) $\sum \frac{3^n}{2^{n+1}+1}$ (c) $\sum \frac{n^n}{n!}$

Exercise 10.9.4. For each of the following power series, determine the interval of convergence. Work through only as many of the details as needed to convince yourself that you are correct, the goal is to be able to identify the interval without a page worth of writing.

1. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ $|x-2|<1$ $[1, 3)$

2. $\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{n}$ $|x-2|<\frac{1}{2}$ $[\frac{3}{2}, \frac{5}{2})$

3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n}$ $|x-2|<2$ $[0, 4)$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{n}$ $|x-2|<1$ $(1, 3]$

5. $\sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$ $|x|<1$ $[-1, 1]$

6. $\sum_{n=1}^{\infty} \frac{nx^n}{n!}$ $(-\infty, \infty)$

7. $\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$ $(-\infty, \infty)$

8. $\sum_{n=1}^{\infty} \frac{\sqrt{n}(x+3)^n}{n^2+1}$ $|x+3|<1$ $[-4, -2]$

9. $\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n}}$ $|x+3|<1$ $[-4, -2)$

10. $\sum_{n=1}^{\infty} n(x-4)^n$ $|x-4|<1$ $(3, 5)$