10.10 Taylor Series

In the previous sections we were able to leverage what we knew about the geometric power series into power series representations for a number of related functions by substitution, multiplication, differentiation, and integration.

We were also able to find a series representation for the exponential by solving an Initial Value Problem. In that case, we were able to uniquely determine a function by solving a differential equation subject to initial data. In essence, we used a relationship between the function and its derivative and its value at one point to generate a function.

In this section we discuss finding power series representations for well behaved functions based on their derivatives at one point. It turns out that if we know all of the derivatives of a function at one point, the function is uniquely determined, provided some technical concerns are satisfied. For our purposes, we will ignore the technical concerns. For the functions we are generally interest in, they are satisfied.

Theorem 10.10.1 (Taylor Series Representation). *If $f(x)$ is represented by a power series with center $x = c$ on an interval $|x - c| < R$ with $R > 0$, the the power series is the* **Taylor Series**

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

In the case $c = 0$, we call the series the **Maclaurin Series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Remark. $f^{(n)}$ denotes the nth derivative with $f^{(0)} = f$.

Remark. $T(x)$ may not converge. $T(x)$ may converge, but not to $f(x)$. The thoerem says, if there is a series representation, then it must be the Taylor series, i.e. the series representation is unique. All of the power series we have seen ealier are the Taylor series for the given functions, more properly, they are all Maclaurin series.

Remark. The full strength is often not needed, nor used. Many applications only require the first few terms of the series representation, the Taylor polynomial. We saw one such Taylor polynomial in 171. The linearization of a function at $x = c$ given by $L(x) = f(c) + f'(c)(x - c)$ is exactly the first partial sum of the series, i.e. the first order Taylor polynomial.

Example 10.10.1. Find the Maclaurin series for $\sin x$, then find it's radius of convergence.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$\vdots \qquad \vdots$

so $\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

Example 10.10.2. Find the Maclaurin series for $\cos x$, then find it's radius of convergence.

We could repeat the argument for $\cos x$, but since series are unique, we can simply differentiate the series above

$$\cos x = \frac{d}{dx} \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x$$

Example 10.10.3. Find the first five non-zero terms in the Taylor series for $f(x) = (x+1)^{3/2}$ about $x = 3$.

$$f(x) = (x+1)^{3/2} \qquad f'(3) = 8$$

$$f'(x) = \frac{3}{2}(x+1)^{1/2} \qquad f'(3) = 3$$

$$f''(x) = \frac{3}{2}\cdot\frac{1}{2}(x+1)^{-1/2} \qquad f''(3) = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$$

$$f'''(x) = \frac{-3}{8}(x+1)^{-3/2} \qquad f'''(3) = \frac{-3}{8}\cdot\frac{1}{8} = \frac{-3}{64}$$

$$f^{(iv)}(x) = \frac{9}{16}(x+1)^{-5/2} \qquad f^{(iv)}(3) = \frac{9}{16}\cdot\frac{1}{32} = \frac{9}{512}$$

$$\vdots$$

so

$$f(x) = (x+1)^{3/2} = 8 + 3(x-3) + \frac{3}{8\cdot 2!}(x-3)^2$$

$$- \frac{3}{64\cdot 3!}(x-3)^3 + \frac{9}{512\cdot 4!}(x-3)^4 + \cdots$$

Maclaurin series of some common functions

Values of x where series converge is indicated in each case

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{all } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad |x| < 1\,,\, x = 1$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x| \le 1$$

This table will be provided, in on form or another, for quizzes and the final exam. For now, we would like to put a few series to good use.

Example 10.10.4. Integrate $\displaystyle\int \frac{\sin x}{x}\, dx$.

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \quad \text{so} \quad \int \frac{\sin x}{x}\, dx$$

$$= A + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

We can also use the series representation to evaluate limits. Use series to compute the following limits.

$$\lim_{x\to 0}\frac{\sin x - x + x^3/6}{2x^5} = \lim_{x\to 0}\frac{\left[x - x^3/6 + x^5/5! \cdots\right] - x + x^3/6}{2x^5}$$

$$= \lim_{x\to 0}\frac{x^5/5! - \cdots}{2x^5} = \lim_{x\to 0}\left(\frac{1}{2\cdot 5!} - \frac{x^2}{2\cdot 7!} + \cdots\right)$$

$$= \frac{1}{2\cdot 5!}$$

$$\lim_{x\to 0}\frac{x\cos x}{1-e^x} = \lim_{x\to 0}\frac{x\left(1 - \frac{x^2}{2!} + \cdots\right)}{1 - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}$$

$$= \lim_{x\to 0}\frac{x - \frac{x^3}{2!} + \cdots}{-x - \frac{x^2}{2!} - \cdots}$$

$$= \lim_{x\to 0}\frac{1 - \frac{x^2}{2!} + \cdots}{-1 - \frac{x}{2!} - \cdots} = -1$$

Although we have only considered power series in the real numbers, they extend to the complex numbers without much change.

Define $i = \sqrt{-1}$, so that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \ldots$.

Exercise 10.10.5. Evaluate $e^{i\theta}$.

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots$$

Since power series are absolutely convergent, we can rearrange

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

so

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= -1$$

or ...

This brings us to one of the most elegant statements in mathematics, Euler's identity.

$$e^{i\pi} + 1 = 0$$

This seems as good of a place as any to end the semester.

Homework

Exercise 10.10.6. Find the Maclaurin series for the following functions and determine the interval of convergence.

1. $f(x) = \dfrac{4x}{2+x}$
2. $g(x) = \dfrac{\sin 2x}{x}$
3. $h(x) = xe^{x^2}$
4. $k(x) = \dfrac{5}{1+x^2}$
5. $f(x) = \sin 3x$
6. $g(x) = e^{x+1}$

Exercise 10.10.7. Find the first four nonzero terms of the Taylor series of the following centered at c.

1. $f(x) = (5-x)^{3/2};\ c = 1$
2. $g(x) = \sin 2x;\ c = \frac{\pi}{4}$

Exercise 10.10.8. Use series to compute the following limits.

1. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$
2. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3 \cos x}$

Exercise 10.10.9. Use an appropriate Maclaurin series to integrate the following.

1. $\displaystyle\int e^{x^4}\, dx$
2. $\displaystyle\int \sin\left(x^2\right) dx$

Exercise 10.10.10. Evaluate the following series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$ $= \cos 2\pi = 1$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!}$ $= e^{\ln 2} = 2$