Ignore the page numbers, these solutions are cobbled together from a couple sources.

1\. A. $\displaystyle\int_{-\infty}^{0} xe^x\,dx = xe^x\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^x\,dx = \left(xe^x - e^x\right)\Big|_{-\infty}^{0} = \lim_{R\to-\infty}\left(xe^x - e^x\right)\Big|_{R}^{0}$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$= \lim_{R\to-\infty}\left(0 - 1 - Re^R + e^R\right)$$

$$= -1 - \lim_{R\to-\infty}\left(Re^R\right)$$

$(-\infty)(0)$ form which is indeterminate, so we use L'Hopital's Rule

$$= -1 - \lim_{R\to-\infty}\left(\frac{R}{e^{-R}}\right)$$

$\frac{(-\infty)}{\infty}$ form, so L'H applies

$$\overset{L'H}{=} -1 - \lim_{R\to-\infty}\left(\frac{1}{-e^{-R}}\right) = -1$$

B. $\displaystyle\int_{4}^{\infty} \frac{5}{x^2-x-6}\,dx = \int_{4}^{\infty}\left(\frac{1}{x-3} - \frac{1}{x+2}\right)dx = \lim_{R\to\infty}\left(\ln|x-3| - \ln|x+2|\right)\Big|_{4}^{R}$

$$\frac{5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$= \lim_{R\to\infty}\left(\underbrace{\ln|R-3| - \ln|R+2|}_{\infty\ -\ \infty\ \text{indeterminate form}} - \ln|4-3| + \ln|4+2|\right)$$

$5 = A(x+2) + B(x-3)$

$$= \lim_{R\to\infty} \ln\left|\frac{R-3}{R+2}\right| + \ln 6$$

Let $x = 3$, $5 = 5A \Rightarrow A = 1$

$x = -2$, $5 = -5B \Rightarrow B = -1$

$$= \ln\left(\lim_{R\to\infty}\frac{R-3}{R+2}\right) + \ln 6 = \ln 6$$

Since $\ln(\cdot)$ is continuous, the limit & the function can switch.

1. C. $\displaystyle\int_0^\infty \frac{2}{3x+5}\,dx = \lim_{R\to\infty} \frac{2}{3}\ln|3x+5|\Big|_0^R = \lim_{R\to\infty} \frac{2}{3}\ln|3R+5| - \frac{2}{3}\ln 5 = \infty$.

so the integral diverges.

D. $\displaystyle\int_0^\infty \frac{\arctan x}{x^2+1}\,dx = \lim_{R\to\infty} \frac{1}{2}\arctan^2 x\Big|_0^R = \lim_{R\to\infty} \frac{1}{2}\arctan^2 R - 0 = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$

E. $\displaystyle\int_0^1 \ln x\,dx = x\ln x\Big|_0^1 - \int_0^1 dx = \lim_{R\to 0^+} \left(x\ln x - x\right)\Big|_R^1 = \lim_{R\to 0^+}\left(-1 - R\ln R + R\right)$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$\displaystyle = -1 - \lim_{R\to 0^+}(R\ln R) \longleftarrow 0\cdot(-\infty)$ indeterminate form

$\displaystyle = -1 - \lim_{R\to 0^+}\left(\frac{\ln R}{1/R}\right) \longleftarrow \frac{-\infty}{\infty}$ indeterminate form,

L'H applies

$\displaystyle \overset{L'H}{=} -1 - \lim_{R\to 0^+}\left(\frac{1/R}{-1/R^2}\right) = -1 - \lim_{R\to 0^+}(-R) = -1$

F. $\displaystyle\int_{-5}^{2} \frac{dx}{\sqrt[5]{x-2}} = \int_{-7}^{0} u^{-1/5}\,du = \lim_{R\to 0^-} \frac{5}{4}u^{4/5}\Big|_{-7}^{R} = \lim_{R\to 0^-}\left(\frac{5}{4}(R)^{4/5} - \frac{5}{4}(-7)^{4/5}\right)$

Let $u = x-2$

$du = dx$

$x=-5 \mapsto u=-7$

$x\to 2 \mapsto u\to 0$

$\displaystyle = -\frac{5}{4}(-7)^{4/5}$

2. A. $\displaystyle\int_1^{\infty} \frac{dx}{x^4+x+7}$

We have $0 \le \dfrac{1}{x^4+x+7} \le \dfrac{1}{x^4}$ for $x \ge 1$.

Additionally, the p-integral $\displaystyle\int_1^{\infty} \frac{dx}{x^4}$ converges since $p = 4 > 1$.

So, by the Comparison Theorem, $\displaystyle\int_1^{\infty} \frac{dx}{x^4+x+7}$ also converges.

B. $\displaystyle\int_2^{\infty} \frac{2}{x-\sqrt{x}}\,dx$

Since $0 \le \dfrac{2}{x} \le \dfrac{2}{x-\sqrt{x}}$ for $x \ge 2$ & $\displaystyle\int_2^{\infty} \frac{2}{x}\,dx = 2\int_2^{\infty} \frac{dx}{x}$ is a divergent p-integral $(p = 1 \le 1)$, by comparison, $\displaystyle\int_2^{\infty} \frac{2}{x-\sqrt{x}}\,dx$ also diverges.

C. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges by comparison since $0 \le \dfrac{\cos x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$ for $x \in (0, \frac{\pi}{2}]$ and $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is a convergent p-integral $(p = \frac{1}{2} < 1)$

__Note:__ in the three arguments above, I wrote them in different orders. The important thing is that each contained three parts.

1) A comparison between __functions__ (not integrals), e.g. $0 \le \dfrac{\cos x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$ for $x \in (0, \frac{\pi}{2}]$

2) Justification that an __integral__ (not function) converges/diverges, e.g. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is a convergent p-integral $(p = \frac{1}{2} < 1)$ ← This will almost always reference a p-integral.

3) A conclusion about an __integral__ (not a function) e.g. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges by comparison.

5.D. $\displaystyle\int_0^2 \frac{e^{x^2}}{x^2}\,dx$

Initially note that for $x \in (0,2]$ $\quad 0 \le \dfrac{1}{x^2} \le \dfrac{e^{x^2}}{x^2}$.

Additionally, the p-integral $\displaystyle\int_0^2 \frac{dx}{x^2}$ diverges $(p = 2 \geq 1)$.

So, by the Comparison Theorem $\displaystyle\int_0^2 \frac{e^{x^2}}{x^2}\,dx$ also diverges.

E. $\displaystyle\int_2^\infty \frac{e^{-x^2}}{x^2+1}\,dx$

Since $e^{-x^2} \le 1$ for all x, we have $0 \le \dfrac{e^{-x^2}}{x^2+1} \le \dfrac{1}{x^2}$.

Now, the p-integral $\displaystyle\int_2^\infty \frac{dx}{x^2}$ converges since $p = 2 > 1$,

so by the Comparison Theorem $\displaystyle\int_2^\infty \frac{e^{-x^2}}{x^2+1}\,dx$ also converges.

F. $\displaystyle\int_0^2 \frac{dx}{x^2+\sqrt{x}}$

For $x \in (0,2]$ we have $0 \le \dfrac{1}{x^2+\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$. Since $\displaystyle\int_0^2 \frac{dx}{\sqrt{x}}$ is a convergent p-integral, $(p = \tfrac{1}{2} < 1)$, by the Comparison Theorem $\displaystyle\int_0^2 \frac{dx}{x^2+\sqrt{x}}$ also converges.

3\. A. $y = 1 + 3x \quad [0, 2]$

$y' = 3$

$1 + (y')^2 = 10$

$$S = \int_0^2 \sqrt{10}\, dx = 2\sqrt{10}$$

B. $y = \frac{1}{3}(2 + x^2)^{3/2} \quad [0,2]$

$y' = x(2 + x^2)^{1/2}$

$1 + (y')^2 = 1 + 2x^2 + x^4$

$= (1 + x^2)^2$

$$S = \int_0^2 \sqrt{(1+x^2)^2}\, dx = \int_0^2 (1 + x^2)\, dx = x + \frac{x^3}{3}\Bigg|_0^2 = 2 + \frac{8}{3} = \frac{14}{3}$$

C. $y = 1 + x^2 \quad [0,2]$

$y' = 2x$

$1 + (y')^2 = 1 + 4x^2$

$$S = \int_0^2 \sqrt{1 + 4x^2}\, dx = \int_0^\varphi \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\right)\Bigg|_0^\varphi$$

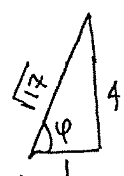
given

let $2x = \tan\theta$

$2dx = \sec^2\theta\, d\theta$

$x = 0 \longmapsto \theta = 0$

$x = 2 \longmapsto \underbrace{\theta = \arctan 4}_{= \varphi}$



so

$\sec\varphi = \sqrt{17}$

$\tan\varphi = 4$

$$= \frac{1}{2}\left(4\sqrt{17} + \ln\left|\sqrt{17} + 4\right|\right)$$

Use a Δ to unwrap $\sec(\arctan 4)$

give the angle a name so we don't have to keep writing $\arctan 4$

D. $y = \frac{x^4}{32} + \frac{1}{x^2} \quad [1,2]$

$y' = \frac{x^3}{8} - \frac{2}{x^3}$

$1 + (y')^2 = 1 + \left(\frac{x^6}{64} - \frac{1}{2} + \frac{4}{x^6}\right)$

$= \frac{x^6}{64} + \frac{1}{2} + \frac{4}{x^6}$

$= \left(\frac{x^3}{8} + \frac{2}{x^3}\right)^2$

$$S = \int_1^2 \sqrt{\left(\tfrac{x^3}{8} + \tfrac{2}{x^3}\right)^2}\, dx = \int_1^2 \left(\frac{1}{8}x^3 + 2x^{-3}\right) dx$$

$$= \frac{x^4}{32} - \frac{1}{x^2}\Bigg|_1^2$$

$$= \left(\frac{16}{32} - \frac{1}{4}\right) - \left(\frac{1}{32} - 1\right)$$

3.E. $y = \ln \cos x \qquad [0, \pi/3]$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$1 + (y')^2 = 1 + \tan^2 x$$
$$= \sec^2 x$$

$$S = \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/3} \sec x \, dx$$

given

$$= \ln |\sec x + \tan x| \Big|_0^{\pi/3}$$

$$= \ln |2 + \sqrt{3}| - \ln |1 + 0|$$

$$= \ln |2 + \sqrt{3}|$$

F. $y = \frac{x^5}{20} + \frac{1}{3x^3} \qquad [1,2]$

$$y' = \frac{x^4}{4} - \frac{1}{x^4}$$

$$1 + (y')^2 = 1 + \left(\frac{x^8}{16} - \frac{1}{2} + \frac{1}{x^8} \right)$$

$$= \frac{x^8}{16} + \frac{1}{2} + \frac{1}{x^8} = \left(\frac{x^4}{4} + \frac{1}{x^4} \right)^2$$

$$S = \int_1^2 \sqrt{\left(x^4/4 + x^{-4} \right)^2}\, dx = \int_1^2 \left(\frac{1}{4} x^4 + x^{-4} \right) dx$$

$$= \frac{x^5}{20} - \frac{1}{3x^3} \Big|_1^2$$

$$= \left(\frac{32}{20} - \frac{1}{24} \right) - \left(\frac{1}{20} - \frac{1}{3} \right)$$

9.A. $y = x^3 \qquad [0,1]$

$$y' = 3x^2$$

$$1 + (y')^2 = 1 + 9x^4$$

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx = 2\pi \cdot \frac{1}{36} \int_1^{10} u^{1/2}\, du$$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$x = 0 \longmapsto u = 1$

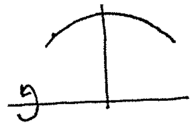
$x = 1 \longmapsto u = 10$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \frac{\pi}{27} \left(10^{3/2} - 1 \right)$$

14

4.B. $y = \sqrt{4-x^2}$ $\quad [-1, 1]$



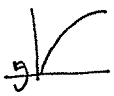
$$y' = \frac{-x}{\sqrt{4-x^2}}$$

$$1+(y')^2 = 1 + \frac{x^2}{4-x^2}$$

$$= \frac{4-x^2+x^2}{4-x^2} = \frac{4}{4-x^2}$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\,dx = 2\pi \int_{-1}^{1} 2\,dx = 4\pi x \Big|_{-1}^{1} = 8\pi$$

C. $y = \sqrt{x}$ $\quad [0,1]$



$$y' = \frac{1}{2\sqrt{x}}$$

$$1+(y')^2 = 1 + \frac{1}{4x}$$

$$= \frac{4x+1}{4x}$$

$$S = 2\pi \int_0^1 \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx = \pi \int_0^1 \sqrt{4x+1}\,dx = \frac{\pi}{4}\int_1^5 u^{1/2}\,du$$

$u = 4x+1$

$du = 4\,dx$

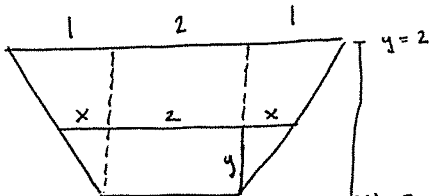
$x=0 \longmapsto u=1$

$x=1 \longmapsto u=5$

$$= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^5$$

$$= \frac{\pi}{6}\left(5^{3/2}-1\right)$$

5.



By similar triangles $\frac{x}{y} = \frac{1}{2}$ so $x = \frac{1}{2}y$.

The width of a slice is $w = 2+2x = 2+y$.

The area of a slice is $(2+y)\,\Delta y$.

The depth of a slice is $(2-y)$.

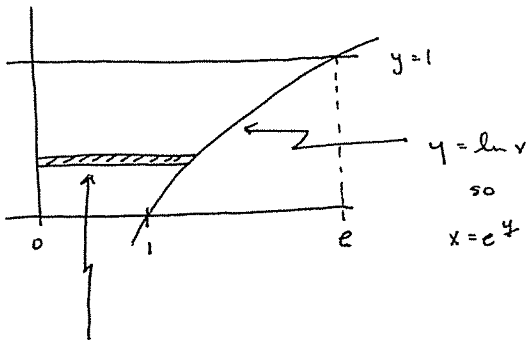
$$\text{Force} = \rho g \int_0^2 (2+y)(2-y)\,dy$$

$$= \rho g \int_0^2 (4-y^2)\,dy$$

$$= \rho g \left(4y - \frac{y^3}{3}\right)\Big|_0^2$$

$$= \frac{16}{3}\rho g$$

6.



The area of this slice is $x \, \Delta y = e^y \, \Delta y$.

The depth of this slice is $(1-y)$.

$$\text{Force} = \rho g \int_0^1 (1-y) e^y \, dy$$

$u = (1-y) \qquad dv = e^y dy$

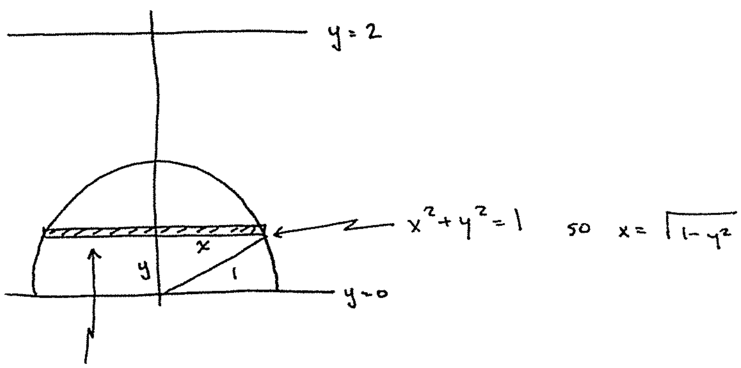
$du = -dy \qquad v = e^y$

$$= \rho g \left[(1-y)e^y \Big|_0^1 + \int_0^1 e^y \, dy \right]$$

$$= \rho g \left(0 - 1 + e^y \Big|_0^1 \right)$$

$$= \rho g \left(0 - 1 + e - 1 \right) = \rho g (e - 2)$$

~~8.~~ 7.



The area of this slice is $2x \, \Delta y = 2\sqrt{1-y^2} \, \Delta y$.

The depth of this slice is $(2-y)$.

This is the area of a quarter circle of radius 1, no need to do a trig sub.

$$\text{Force} = \rho g \int_0^1 2(2-y)\sqrt{1-y^2} \, dy = 4\rho g \int_0^1 \sqrt{1-y^2} \, dy - 2\rho g \int_0^1 y\sqrt{1-y^2} \, dy$$

$u = 1 - y^2$

$du = -2y \, dy$

$y = 0 \mapsto u = 1$

$y = 1 \mapsto u = 0$

$$= 4\rho g \left(\frac{\pi}{4} \right) + \rho g \int_1^0 u^{1/2} \, du$$

$$= 4\rho g \left(\frac{\pi}{4} \right) + \rho g \cdot \frac{2}{3} u^{3/2} \Big|_1^0$$

$$= \rho g \pi - \frac{2}{3} \rho g = \rho g \left(\pi - \frac{2}{3} \right)$$

Q. $\displaystyle\int \frac{e^{2x}}{e^{2x}+5e^x+6}\,dx = \int \frac{u}{u^2+5u+6}\,du = \int \left(\frac{3}{u+3} - \frac{2}{u+2}\right) du$

Let $u = e^x$

$du = e^x\,dx$

$$\frac{u}{(u+3)(u+2)} = \frac{A}{u+3} + \frac{B}{u+2}$$

$$= 3\ln(e^x+3) - 2\ln(e^x+2) + C$$

$$u = A(u+2) + B(u+3)$$

let $u=-2$: $-2 = B$

$u=-3$: $-3 = -A \Rightarrow A = 3$

R. $\displaystyle\int_0^{\pi/2} \cos x\sqrt{1+3\sin^2 x}\,dx = \frac{1}{\sqrt{3}}\int_0^{\pi/3} \sec^3\theta\,d\theta = \frac{1}{2\sqrt{3}}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/3}$

Let $\tan\theta = \sqrt{3}\sin x$

$\sec^2\theta\,d\theta = \sqrt{3}\cos x\,dx$

$$= \frac{1}{2\sqrt{3}}\left(2\cdot\sqrt{3} + \ln|2+\sqrt{3}|\right)$$

x	$\longmapsto$	θ
0		0
$\pi/2$		$\pi/3$

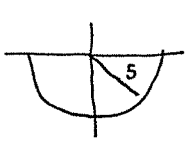
$\boxed{\text{this is for 13.F.}}$

8. A $\quad x = t+1$, $y = t^3+1$ so $x-1 = t$ so $y = (x-1)^3+1$

B. $x = e^{2t}+1$, $y = e^{6t}+1$ so $x-1 = e^{2t}$ so $y = (x-1)^3+1$

C. $x = \dfrac{2+t}{1+t} = 1 + \dfrac{1}{1+t}$, $y = \dfrac{1}{(1+t)^3}+1$ so $x-1 = \dfrac{1}{1+t}$ so $y = (x-1)^3+1$

D. $x = 5\cos t$, $y = 5\sin t$, $t \in [\pi, 2\pi]$



so $y = -\sqrt{25-x^2}$

9. A. $x = 4t+1$ $\quad x' = 4$

$y = 3t-1$ $\quad y' = 3$

$0 \le t \le 2$

so $ds = \sqrt{(x')^2+(y')^2}\,dt = 5\,dt$

$$S = \int_0^2 5\,dt = 10$$

B. $x = 3t^2+3$ $\quad x' = 6t$

$y = t^3-2$ $\quad y' = 3t^2$

$0 \le t \le \sqrt{5}$

$(x')^2+(y')^2 = 36t^2+9t^4$

$ds = \sqrt{36t^2+9t^4}\,dt$

$= 3t\sqrt{4+t^2}\,dt$

$$S = \int_0^{\sqrt5} 3t\sqrt{4+t^2}\,dt = \frac{3}{2}\int_4^9 u^{1/2}\,du$$

$u = 4+t^2$

$du = 2t\,dt$

t	$\mapsto$ u
0	4
$\sqrt5$	9

$$= u^{3/2}\Big|_4^9 = 27-8$$

$$= 19$$

C. $x = e^{2t}\cos t$

$y = e^{2t}\sin t$ $\qquad$ see 7.E.

$t \in [0,\pi]$

This is a cycloid.

D. $x = t - \sin t$ $\quad x' = 1-\cos t$

$y = 1-\cos t$ $\quad y' = \sin t$

$t \in [0, 2\pi]$

$(x')^2+(y')^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t$

$= 2 - 2\cos t$

$= 2(1-\cos t)$ $\longleftarrow$ $1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$

$= 4\sin^2 \frac{t}{2}$ $\qquad$ so $ds = 2\sin\frac{t}{2}\,dt$

$$S = \int_0^{2\pi} ds = \int_0^{2\pi} 2\sin\tfrac{t}{2}\,dt = -4\cos\left(\tfrac{t}{2}\right)\Big|_0^{2\pi} = -4\,(-1-1) = 8$$

10. A. $x = 2\sin t$ $\quad x' = 2\cos t$ $\quad x'(\pi/6) = \sqrt3$

$y = t$ $\quad y' = 1$ $\quad y'(\pi/6) = 1$

so $\left.\dfrac{dy}{dx}\right|_{t=\pi/6} = \dfrac{1}{\sqrt3}$ & $\left.\dfrac{ds}{dt}\right|_{t=\pi/6} = \sqrt{(\sqrt3)^2+1^2} = 2$

B. $x = \dfrac{t+2}{4}$ $\quad x' = \dfrac14$

$y = \dfrac{t-3}{3}$ $\quad y' = \dfrac13$

so $\left.\dfrac{dy}{dx}\right|_{t=\frac{-\pi 3}{e^{37}}} = \dfrac{1/3}{1/4} = 4/3$ & $\left.\dfrac{ds}{dt}\right|_{t=\text{ugly number}} = \sqrt{\dfrac{1}{16}+\dfrac19} = \dfrac{5}{12}$

10. C. $x = 2\cos 2t$ $\quad x' = -4\sin 2t$ $\quad x'\left(\frac{3\pi}{8}\right) = -4\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y = \sin 2t$ $\quad y' = 2\cos 2t$ $\quad y'\left(\frac{3\pi}{8}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

So $\left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{8}} = \frac{-\sqrt{2}}{-2\sqrt{2}} = \frac{1}{2}$ $\qquad \left.\frac{ds}{dt}\right|_{t=\frac{3\pi}{8}} = \sqrt{(-2\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{10}$

D. $x = \frac{t}{t+1}$ $\quad x' = \frac{1}{(t+1)^2}$ $\quad x'(1) = \frac{1}{4}$

$y = t^2$ $\quad y' = 2t$ $\quad y'(1) = 2$

so $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{1/4} = 8$ $\qquad \left.\frac{ds}{dt}\right|_{t=1} = \sqrt{\frac{1}{16} + 4} = \frac{\sqrt{65}}{4}$

11. A. $r = 2$ so $x^2 + y^2 = 4$

B. $r = \frac{3}{\cos\theta}$ so $r\cos\theta = 3$ or $x = 3$

C. $r = 2\csc\theta$ so $r\sin\theta = 2$ or $y = 2$

D. $r = \frac{6}{2\cos\theta - 3\sin\theta}$ so $2r\cos\theta - 3r\sin\theta = 6$ or $2x - 3y = 6$

E. $r = -2\sin\theta$ so $r^2 = -2r\sin\theta$ or $x^2 + y^2 = -2y$

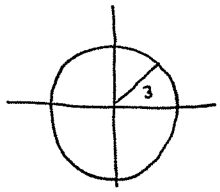
or if you want to be fancy, ...

$x^2 + (y+1)^2 = 1$

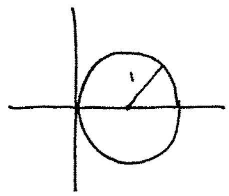
F. $\theta = \frac{\pi}{4}$ or $y = x$

12

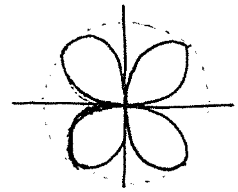
A. $r = 3$



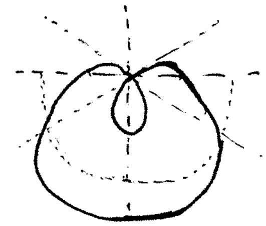
B. $r = 2\cos\theta$



C. $r = \sin 2\theta$



D. $r = 1 - 2\sin\theta$



see 9.c for the upside down version

13

A. $r = 3$

$r' = 0$

so $ds = \sqrt{9+0}\, d\theta = 3\, d\theta$

$$S = \int_0^{2\pi} 3\, d\theta = 6\pi$$

B. $r = \theta$

$\frac{dr}{d\theta} = r' = 1$

so $ds = \sqrt{\theta^2 + 1}\, d\theta$

so $$S = \int_0^{\pi} \sqrt{\theta^2+1}\, d\theta = \int_0^{\arctan \pi} \sec^3\phi\, d\phi$$

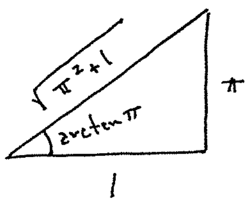
Let $\theta = \tan\phi$

$d\theta = \sec^2\phi\, d\phi$

θ	ϕ
0	0
π	$\arctan(\pi)$

$$= \frac{1}{2}\left(\sec\phi\tan\phi + \ln|\sec\phi + \tan\phi|\right)\Bigg|_0^{\arctan(\pi)}$$

$$= \frac{1}{2}\left(\pi\sqrt{\pi^2+1} + \ln\left(\pi + \sqrt{\pi^2+1}\right)\right)$$



so $\sec(\arctan\pi) = \sqrt{\pi^2+1}$

C. $r = \theta^2$

$r' = 2\theta$

so $ds = \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \theta\sqrt{\theta^2+4}\, d\theta$

so $$S = \int_0^{\pi} \theta\sqrt{\theta^2+4}\, d\theta = \frac{1}{2}\int_4^{\pi^2+4} u^{1/2}\, du = \frac{1}{3}u^{3/2}\Bigg|_4^{\pi^2+4}$$

$u = \theta^2 + 4$

$du = 2\theta\, d\theta$

$$= \frac{1}{3}\left[(\pi^2+4)^{3/2} - 8\right]$$

D. $r = \theta^2 - 1$

$r' = 2\theta$

so $ds = \sqrt{(\theta^2-1)^2 + (2\theta)^2}\, d\theta$

$= \sqrt{\theta^4 - 2\theta^2 + 1 + 4\theta^2}\, d\theta$

$= \sqrt{\theta^4 + 2\theta^2 + 1}\, d\theta$

$= \sqrt{(\theta^2+1)^2}\, d\theta$

so $$S = \int_0^{\pi} (\theta^2+1)\, d\theta = \frac{\theta^3}{3} + \theta\Bigg|_0^{\pi}$$

$$= \frac{\pi^3}{3} + \pi$$

~~8~~. E. $r = e^{2\theta}$

13 $\quad r' = 2e^{2\theta}$ $\quad$ so $ds = \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta = \sqrt{5}\, e^{2\theta}\, d\theta$

$$S = \int_0^{\pi} \sqrt{5}\, e^{2\theta}\, d\theta = \frac{\sqrt{5}}{2} e^{2\theta} \Big|_0^{\pi} = \frac{\sqrt{5}}{2}\left(e^{2\pi} - 1\right)$$

F. $r = \cos^2\theta$

$r' = -2\cos\theta\sin\theta$

$$(r)^2 + (r')^2 = \cos^4\theta + 4\cos^2\theta\sin^2\theta$$
$$= \cos^2\theta\left(\cos^2\theta + 4\sin^2\theta\right)$$
$$= \cos^2\theta\left(1 + 3\sin^2\theta\right)$$

$4\sin^2\theta = \sin^2\theta + 3\sin^2\theta$

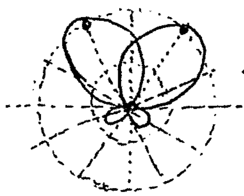
SEE solution before 8.A.

$$\text{so} \quad S = 4\int_0^{\pi/2} \cos\theta\sqrt{1 + 3\sin^2\theta}\, d\theta = \frac{2}{\sqrt{3}}\left(2\sqrt{3} + \ln\left|2 + \sqrt{3}\right|\right)$$

Symmetry

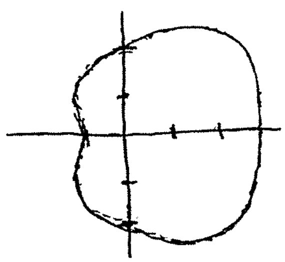
see ~~#1.R~~

14 ~~9~~.



A cute little butterfly! (or see next page for a fancy version)

15 ~~10~~. A. Inside $r = 2 + \cos\theta$

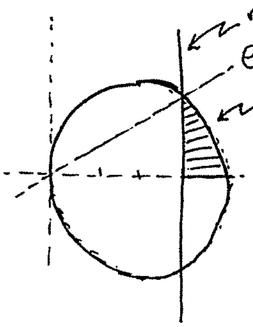


$$\frac{1}{2}\int_0^{2\pi} (2 + \cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(4 + 4\cos\theta + \cos^2\theta\right) d\theta$$

$\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right)$

$$= \frac{1}{2}\left(4\theta + 4\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{9\pi}{2}$$

B.



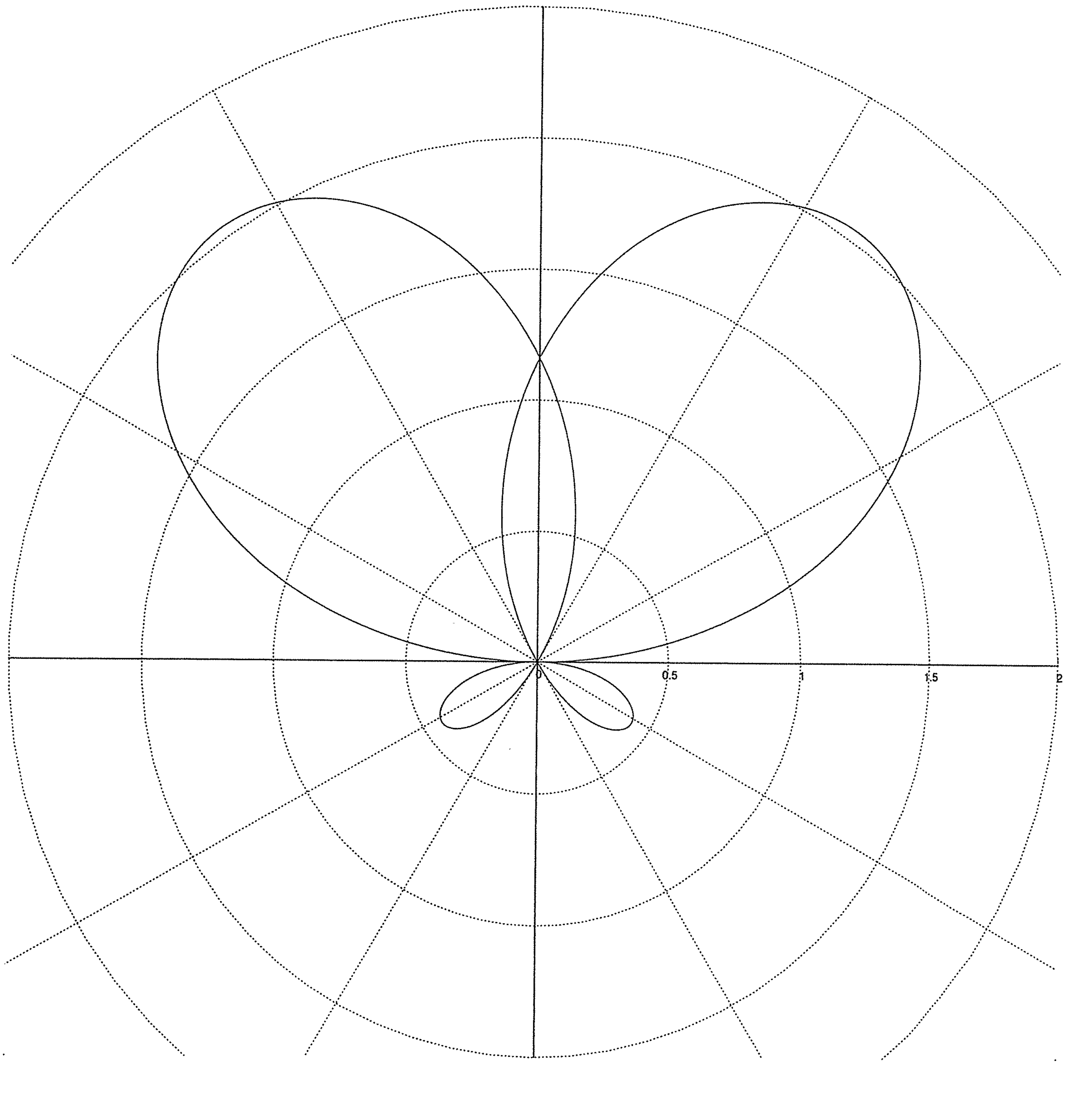
Intersect when

$$3\sec\theta = 4\cos\theta$$
$$\frac{3}{4} = \cos^2\theta$$
$$\frac{\sqrt{3}}{2} = \cos\theta$$
$$\text{so } \theta = \pi/6$$

We consider only the top half, by symmetry

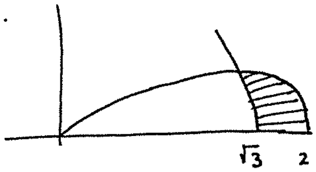
$$2 \cdot \frac{1}{2}\int_0^{\pi/6} \left(4^2\cos^2\theta - 9\sec^2\theta\right) d\theta = \int_0^{\pi/6} \left(8 + 8\cos 2\theta - 9\sec^2\theta\right) d\theta$$

$$= 8\theta + 4\sin 2\theta - 9\tan\theta = \frac{4\pi}{3} + 4\,\frac{\sqrt{3}}{2} - \frac{9}{\sqrt{3}} = \frac{4\pi}{3} - \sqrt{3}$$


0
0.5
1
1.5
2

10

10. A.



$r=\sqrt{3}$ & $r=2\cos 3\theta$ intersect when $\frac{\sqrt{3}}{2}=\cos 3\theta$

so $3\theta = \pi/6$

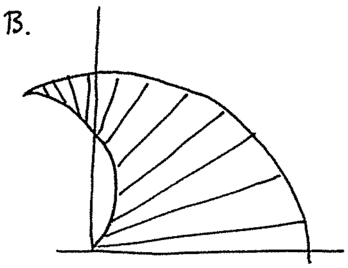
so $\theta = \pi/18$

Symmetry

$$6\cdot\frac{1}{2}\int_0^{\pi/18}\left(4\cos^2 3\theta - 3\right)d\theta = 3\int_0^{\pi/18}\left(2+2\cos 6\theta - 3\right)d\theta = 3\left(-\theta+\frac{1}{3}\sin 6\theta\right)\Big|_0^{\pi/18}$$

$$= 3\left(-\frac{\pi}{18}+\frac{1}{3}\cdot\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}-\frac{\pi}{6}$$

B.



$r=3$ & $r=2-2\cos\theta$ intersect when $-\frac{1}{2}=\cos\theta$

so $\theta = \frac{2\pi}{3}$

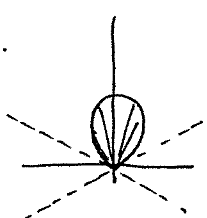
Symmetry

$$2\cdot\frac{1}{2}\int_0^{2\pi/3}\left(9-4+8\cos\theta-4\cos^2\theta\right)d\theta$$

$$= \int_0^{2\pi/3}\left(5+8\cos\theta-2-2\cos 2\theta\right)d\theta$$

$$= 3\theta+8\sin\theta-\sin 2\theta\Big|_0^{2\pi/3} = 2\pi+8\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2} = 2\pi+\frac{9\sqrt{3}}{2}$$

C.



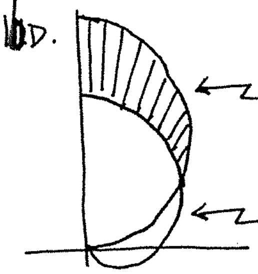
The inner loop happens when $r\le 0$, so $1+2\sin\theta\le 0$ when $\frac{7\pi}{6}\le\theta\le\frac{11\pi}{6}$

$$\frac{1}{2}\int_{7\pi/6}^{11\pi/6}\left(1+2\sin\theta\right)^2 d\theta = \frac{1}{2}\int_{7\pi/6}^{11\pi/6}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta = \frac{1}{2}\int_{7\pi/6}^{11\pi/6}\left(3+4\sin\theta-2\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left(3\theta-4\cos\theta-\sin 2\theta\right)\Big|_{7\pi/6}^{11\pi/6} = \frac{1}{2}\left(2\pi-4\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \pi-\frac{3\sqrt{3}}{2}$$

16D.



$r = 3\sin\theta$

$r = 1 + \sin\theta$

intersect when $3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

so $\theta = \frac{\pi}{6}$

Symmetry

$$2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta) \right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(8\sin^2\theta - 1 - 2\sin\theta \right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(4 - 4\cos 2\theta - 1 - 2\sin\theta \right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(3 - 4\cos 2\theta - 2\sin\theta \right) d\theta = 3\theta - 2\sin 2\theta + 2\cos\theta \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{3\pi}{2} - \left(\frac{\pi}{2} - 2 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{\sqrt{3}}{2} \right)$$

$$= \pi$$