1. Integrate.

(a) \[ 10 \int_0^2 xe^{x^2} \, dx = \frac{1}{2} \int_0^4 e^u \, du = \frac{1}{2} e^u \bigg|_0^4 = \frac{1}{2} \left( e^4 - 1 \right) \]

\( u = x^2 \)

\( du = 2x \, dx \)

\[ \frac{\sqrt{x}}{2} \mapsto \frac{u}{2} \]

\[ 2 \quad 4 \]

(b) \[ 10 \int_0^1 xe^{2x} \, dx = \frac{x}{2} e^{2x} \bigg|_0^1 - \int_0^1 \frac{1}{2} e^{2x} \, dx = \frac{1}{2} e^2 - \frac{1}{4} e^{2x} \bigg|_0^1 \]

\[ = \frac{1}{2} e^2 - \frac{1}{4} \left( e^2 - 1 \right) \]

\[ = \frac{1}{4} e^2 + \frac{1}{4} = \frac{1}{4} \left( e^2 + 1 \right) \]

(c) \[ 10 \int e^{x^2} \, dx = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \, dx \quad \text{[HINT: use a series.]} \]

\[ = A + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} \]
\[ \frac{5x^2 + 3x + 4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 1} \]

\[ 5x^2 + 3x + 4 = A(x^2 + 1) + (Bx + C)(x + 1) \]

\[ \begin{align*}
A & = \frac{5 - 3 + 4}{2} = 3 \\
B & = \frac{5}{A} = 2 \\
C & = \frac{4}{A} = 1
\end{align*} \]

\[ \int \left( \frac{3}{x+1} + \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} \right) \, dx = 3 \ln |x+1| + \ln (x^2 + 1) + \arctan x + C \]
3. \[ \int \sqrt{9 - x^2} \, dx. \]

\[ \begin{align*}
\text{Let } x &= 3 \sin \theta \\
\text{ } dx &= 3 \cos \theta \, d\theta \\
9 - x^2 &= 9 - 9 \sin^2 \theta \\
&= 9 \cos^2 \theta \\
\int \sqrt{9 - x^2} \, dx &= \int 9 \cos^2 \theta \, d\theta \\
&= \frac{9}{2} \int \left( \cos 2\theta + \frac{1}{2} \sin 2\theta \right) \, d\theta \\
&= \frac{9}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{9}{2} \left( \theta + \sin \theta \cos \theta \right) + C \\
&= \frac{9}{2} \left( \arcsin \frac{x}{3} + \frac{x \sqrt{9 - x^2}}{9} \right) + C
\end{align*} \]
4. A tank on the moon is filled with liquid oxygen of density $\rho$. The gravitational constant on the moon is $g$. The exterior of the tank is generated by rotating the curve $y = 4 - x^2$ about the $y$-axis for $y \geq 0$. There is a spout protruding 1 m above the top of the tank.

If the tank is full, express the work required to empty the tank through the spout as an integral. Do not evaluate the integral.

\[
W = \int_0^4 g \rho \pi \left( \text{radius} \right)^2 \Delta y \ (\text{distance})
\]
\[
= \int_0^4 g \rho \pi x^2 (5 - y) \, dy
\]

but $x^2 = 4 - y$, so

\[
W_{\text{work}} = \int_0^4 g \rho \pi (4 - y)(5 - y) \, dy
\]
5. Please circle **True** or **False**, as appropriate.

(a) **T**  **F** : If \( a_n \to 0 \) as \( n \to \infty \), the sequence \( \{a_n\} \) converges.

(b) **T**  **F** : If \( a_n \to 0 \) as \( n \to \infty \), the series \( \sum a_n \) converges.

(c) **T**  **F** : If \( \sum |a_n| \) converges, then \( \sum a_n \) converges.

(d) **T**  **F** : If \( \sum |a_n| \) diverges, then \( \sum a_n \) diverges.

(e) **T**  **F** : If \( \sum a_n(x-4)^n \) diverges for \( x = 5 \) it diverges for \( x = 0 \).

6. Find an example that satisfies each of the given requirements.

(a) A sequence that converges but is not monotone.

\[
\left\{ \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \cdots \right\}
\]

(b) A bounded sequence that diverges.

\[
\{-1, 1, -1, 1, -1, \cdots \}
\]

(c) A series that converges but the Ratio Test is inconclusive.

\[
\sum \frac{1}{n^2}
\]

(d) A series that can be shown to diverge by the Test for Divergence.

\[
\sum n
\]

(e) A series that can be shown to converge by the Integral Test but no other test we discussed.

\[
\sum \frac{1}{n \left( \ln n \right)^2}
\]
7. When using the Direct comparison test...

(a) \(2\) which of the following inequalities is useful to show \(\sum \frac{2}{n+5}\) diverges?

i. \(0 < \frac{2}{n+5} < \frac{2}{n}\)  ii. \(0 < \frac{1}{n} < \frac{2}{n+5}\)  iii. Neither  iv. Both

(b) \(2\) which of the following inequalities is useful to show \(\sum \frac{2}{3n^2 - 1}\) converges?

i. \(0 < \frac{2}{3n^2 - 1} < \frac{1}{n^2}\)  ii. \(0 < \frac{2}{3n^2} < \frac{2}{3n^2 - 1}\)  iii. Neither  iv. Both

8. \(16\) Show the following series converges conditionally. Provide appropriate justification, i.e. explicitly write out the appropriate tests, draw appropriate conclusions, and use appropriate notation.

\[
\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{2^n(2n-1)}
\]

Root Test

\[
\sqrt[n]{\frac{(-1)^n(x-2)^n}{2^n(2n-1)}} \quad n \to \infty
\]

\[
\frac{|x-2|}{2} < 1 \quad \text{i.e.} \quad |x-2| < 2
\]

\[
\sum_{n=1}^{\infty} \frac{1}{2n-1}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(2n-1)} = \sum_{n=1}^{\infty} \frac{1}{2n-1} \quad \text{which diverges by comparison since}
\]

\[
0 < \frac{1}{2n} < \frac{1}{2n-1} \quad \text{and} \sum \frac{1}{2^n} \text{ diverges}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(2n-1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \quad \text{converges by the A.S.T. since}
\]

i) \(\frac{1}{2n-1} > 0\),  ii) \(\frac{1}{2n-1} > \frac{1}{2n+1}\), and  iii) \(\frac{1}{2^n-1} \to 0\)
No work or formal justification is expected on this page.

9. [10] For the following series, specify what series you would compare each to and based on your comparison, decide if it converges or diverges.

(a) \( \sum_{n=2}^{\infty} \frac{n + 1}{\sqrt{n^5 + 4n + 1}} \) compare to \( \sum_{n=2}^{\infty} \frac{n}{n^{5/2}} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} \) so it **CONVERGES** / DIVERGES

(b) \( \sum_{n=2}^{\infty} \frac{2^n}{n3^n} \) compare to \( \sum_{n=2}^{\infty} \left( \frac{2}{3} \right)^n \) so it **CONVERGES** / DIVERGES

(c) \( \sum_{n=2}^{\infty} \frac{\sqrt{1 + n}}{n + 1} \) compare to \( \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}} \) so it **CONVERGES** / DIVERGES

(d) \( \sum_{n=2}^{\infty} \frac{4^n \sqrt{n}}{3^n} \) compare to \( \sum_{n=2}^{\infty} \left( \frac{4}{3} \right)^n \) so it **CONVERGES** / DIVERGES

(e) \( \sum_{n=2}^{\infty} \frac{71}{n^2 + \sqrt{n}} \) compare to \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) so it **CONVERGES** / DIVERGES

10. [10] Find the **Interval of Convergence** of the following power series.

(a) \( \sum_{n=0}^{\infty} \frac{(x - 2)^n}{3^n} \) \( |x - 2| < 3 \) \((-1, 5)\)

(b) \( \sum_{n=0}^{\infty} \frac{(x - 5)^n}{(n + 17)!} \) \((-\infty, \infty)\)

(c) \( \sum_{n=0}^{\infty} \frac{(x + 1)^n}{n2^n} \) \( |x + 1| < 2 \) \([-3, 1)\)
11. Find a power series for \( f(x) = \frac{4}{2 + x} \) centered at \( x = 2 \) and where it is valid.

\[
\frac{4}{2 + (x - 2) + 2} = \frac{4}{4 + (x - 2)} = \frac{1}{1 - \left(\frac{x - 2}{4}\right)} = \sum_{n=0}^{\infty} \left(\frac{x - 2}{-4}\right)^n
\]

\[
\left|\frac{x - 2}{-4}\right| < 1
\]

\[
\text{i.e. } |x - 2| < 4
\]

12. Consider a function \( f \) with Maclaurin series

\[
f(x) = 1 - \frac{x}{5} + \frac{x^3}{25} - \frac{x^5}{50} + \cdots
\]

and a function \( g \) defined by

\[
g(x) = \int f(x) \, dx \quad \text{with} \quad g(0) = 2.
\]

(a) Find the first four nonzero terms of the Maclaurin series for \( g \).

\[
g(x) = 1 + x - \frac{x^2}{10} + \frac{x^4}{100} - 
\]

\[
g(0) = 2 \quad \text{so} \quad A = 2
\]

\[
g(x) = 2 + x - \frac{x^2}{10} + \frac{x^4}{100} - 
\]

(b) While we do not know \( g \) explicitly, we do know the first few terms. Let \( G_3(x) \) be the first three nonzero terms of \( g \) you found above. Evaluate \( G_3(0.1) \).

\[
G_3(x) = 2 + x - \frac{x^2}{10} \quad \text{so} \quad G_3(0.1) = 2 + 0.1 - 0.001 = 2.099
\]

(c) Find an error estimate, i.e. \( |g(0.1) - G_3(0.1)| < \)