Math 172 | Exam 3 | Name: ______________

22 June 2016 | Reasonably Simplify | Point Values in $\boxed{\text{boxes}}$.

1. Integrate.

(a) $\boxed{10}$ $\int_0^2 xe^{x^2}\,dx = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}\left(e^4 - 1\right)$

$u = x^2$

$du = 2x\,dx$

x	$\mapsto$	u
0		0
2		4

(b) $\boxed{10}$ $\int_0^1 xe^{2x}\,dx = \frac{x}{2}e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx = \frac{1}{2}e^2 - \frac{1}{4}e^{2x}\Big|_0^1$

$u = x \qquad dv = e^{2x}\,dx$

$du = dx \qquad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}e^2 - \frac{1}{4}\left(e^2 - 1\right)$

$= \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}\left(e^2 + 1\right)$

(c) $\boxed{10}$ $\int e^{x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx$ [HINT: use a series.]

$= A + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,n!}$

2. $\boxed{15}$ Integrate $\displaystyle\int \frac{5x^2+3x+4}{(x+1)(x^2+1)}\,dx$.

$$\frac{5x^2+3x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$5x^2+3x+4 = A(x^2+1) + (Bx+C)(x+1)$$

<u>Let $x=-1$</u>

$5-3+4 = A(2) \quad \Rightarrow \quad A=3$

<u>x^2</u>

$5 = A+B \quad \Rightarrow \quad B=2$

<u>x^0</u>

$4 = A+C \quad \Rightarrow \quad C=1$

$$\int \left(\frac{3}{x+1} + \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx = 3\ln|x+1| + \ln(x^2+1) + \arctan x + C$$

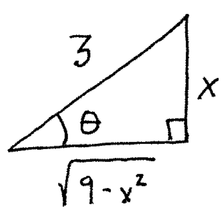
3. $\boxed{15}$ Integrate $\int \sqrt{9 - x^2}\, dx$.

Let $x = 3 \sin \theta$

$dx = 3 \cos \theta \, d\theta$

$9 - x^2 = 9 - 9 \sin^2 \theta$

$= 9 \cos^2 \theta$



$$= \int 9 \cos^2 \theta \, d\theta$$

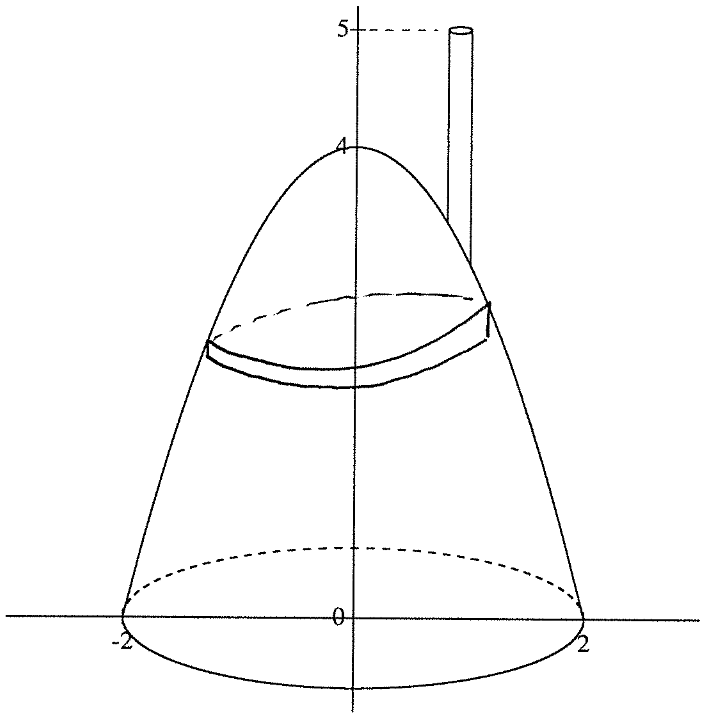
$$= \frac{9}{2} \int (1 + \cos 2\theta) \, d\theta$$

$$= \frac{9}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{9}{2} \left(\theta + \sin \theta \cos \theta \right) + C$$

$$= \frac{9}{2} \left(\arcsin \frac{x}{3} + \frac{x\sqrt{9-x^2}}{9} \right) + C$$

4. $\boxed{10}$ A tank on the moon is filled with liquid oxygen of density ρ. The gravitational constant on the moon is g. The exterior of the tank is generated by rotating the curve $y = 4 - x^2$ about the y-axis for $y \geq 0$. There is a spout protruding 1 m above the top of the tank.



If the tank is full, express the work required to empty the tank through the spout as an integral. **Do not evaluate the integral.**

$$W_i = \rho g \, \pi \, (\text{radius})^2 \Delta y \; (\text{distance})$$

$$= \rho g \pi \, x^2 (5-y) \, dy \qquad \text{but} \quad x^2 = 4-y \text{ , so}$$

$$\text{Work} = \rho g \pi \int_0^4 (4-y)(5-y) \, dy$$

5. $\boxed{10}$ Please circle **True** or **False**, as appropriate.

(a) **T** / **F** : If $a_n \to 0$ as $n \to \infty$, the sequence $\{a_n\}$ converges.

(b) **T** / **F** : If $a_n \to 0$ as $n \to \infty$, the series $\sum a_n$ converges.

(c) **T** / **F**: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

(d) **T** / **F** : If $\sum |a_n|$ diverges, then $\sum a_n$ diverges.

(e) **T** / **F** : If $\sum a_n(x-4)^n$ diverges for $x = 5$ it diverges for $x = 0$.

6. $\boxed{10}$ Find an example that satisfies each of the given requirements.

(a) A sequence that converges but is not monotone.

$$\left\{ 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots \right\}$$

(b) A bounded sequence that diverges.

$$\{ -1, 1, -1, 1, -1, \cdots \}$$

(c) A series that converges but the Ratio Test is inconclusive.

$$\sum \frac{1}{n^2}$$

(d) A series that can be shown to diverge by the Test for Divergence.

$$\sum n$$

(e) A series that can be shown to converge by the Integral Test but no other test we discussed.

$$\sum \frac{1}{n(\ln n)^2}$$

7. When using the Direct comparison test...

(a) $\boxed{2}$ which of the following inequalities is useful to show $\sum \frac{2}{n+5}$ diverges?

i. $0 < \frac{2}{n+5} < \frac{2}{n}$

ii. $0 < \frac{1}{n} < \frac{2}{n+5}$

iii. Neither

iv. Both

(b) $\boxed{2}$ which of the following inequalities is useful to show $\sum \frac{2}{3n^2-1}$ converges?

i. $0 < \frac{2}{3n^2-1} < \frac{1}{n^2}$

ii. $0 < \frac{2}{3n^2} < \frac{2}{3n^2-1}$

iii. Neither

iv. Both

8. $\boxed{16}$ Show the following series converges conditionally. Provide appropriate justification, i.e. explicitly write out the appropriate tests, draw appropriate conclusions, and use appropriate notation.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{2^n(2n-1)}$$

This series goes with a find the interval of convergence question. This was intended to be the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$. Sorry for the mix up.

Root Test

$$\sqrt[n]{\left|\frac{(-1)^n (x-2)^n}{2^n(2n-1)}\right|} \xrightarrow[n\to\infty]{} \frac{|x-2|}{2} < 1, \quad \text{i.e.} \quad |x-2| < 2$$

x = 0

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n(2n-1)} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$ which diverges by comparison ~~to~~ since

$$0 < \frac{1}{2n} < \frac{1}{2n-1} \quad \& \quad \sum \frac{1}{2n} \text{ diverges}$$

x = 4

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n(2n-1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$ converges by the A.S.T. since

i) $\frac{1}{2n-1} > 0$, ii) $\frac{1}{2n-1} > \frac{1}{2n+1}$, and iii) $\frac{1}{2n-1} \xrightarrow[n\to\infty]{} 0$

No work or formal justification is expected on this page.

9. $\boxed{10}$ For the following series, specify what series you would compare each to and based on your comparison, decide if it converges or diverges.

(a) $\sum_{n=2}^{\infty} \frac{n+1}{\sqrt{n^5+4n+1}}$ compare to $\sum \frac{n}{n^{5/2}} = \sum \frac{1}{n^{3/2}}$ so it CONVERGES / DIVERGES

(b) $\sum_{n=2}^{\infty} \frac{2^n}{n3^n}$ compare to $\sum \left(\frac{2}{3}\right)^n$ so it CONVERGES / DIVERGES

(c) $\sum_{n=2}^{\infty} \frac{\sqrt{1+n}}{n+1}$ compare to $\sum \frac{\sqrt{n}}{n} = \sum \frac{1}{n^{1/2}}$ so it CONVERGES / DIVERGES

(d) $\sum_{n=2}^{\infty} \frac{4^n\sqrt{n}}{3^n}$ compare to $\sum \left(\frac{4}{3}\right)^n$ so it CONVERGES / DIVERGES

(e) $\sum_{n=2}^{\infty} \frac{71}{n^2+\sqrt{n}}$ compare to $\sum \frac{1}{n^2}$ so it CONVERGES / DIVERGES

10. $\boxed{10}$ Find the **Interval of Convergence** of the following power series.

(a) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ $|x-2| < 3$ $(-1, 5)$

(b) $\sum_{n=0}^{\infty} \frac{(x-5)^n}{(n+17)!}$ $(-\infty, \infty)$

(c) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n2^n}$ $|x+1| < 2$ $[-3, 1)$

11. $\boxed{10}$ Find a power series for $f(x) = \dfrac{4}{2+x}$ centered at $x = 2$ and where it is valid.

$$f(x) = \frac{4}{2+(x-2)+2} = \frac{4}{4+(x-2)} = \frac{1}{1-\left(\frac{x-2}{-4}\right)} = \sum_{n=0}^{\infty}\left(\frac{x-2}{-4}\right)^n$$

for $\left|\frac{x-2}{-4}\right| < 1$

i.e. $|x-2| < 4$

12. Consider a function f with Maclaurin series

$$f(x) = 1 - \frac{x}{5} + \frac{x^3}{25} - \frac{x^5}{50} + \cdots$$

and a function g defined by

$$g(x) = \int f(x)\,dx \quad \text{with} \quad g(0) = 2.$$

(a) $\boxed{6}$ Find the first **four** nonzero terms of the Maclaurin series for g.

$$g(x) = A + x - \frac{x^2}{10} + \frac{x^4}{100} - \cdots$$

$g(0) = 2$, so $A = 2$

$$g(x) = 2 + x - \frac{x^2}{10} + \frac{x^4}{100} - \cdots$$

(b) $\boxed{2}$ While we do not know g explicitly, we do know the first few terms. Let $G_3(x)$ be the first **three** nonzero terms of g you found above. Evaluate $G_3(0.1)$.

$G_3(x) = 2 + x - \frac{x^2}{10}$ so $G_3(0.1) = 2 + 0.1 - 0.001 = 2.099$

(c) $\boxed{2}$ Find an error estimate, i.e. $|g(0.1) - G_3(0.1)| <$ $\frac{0.1^4}{100} = \frac{1}{10^6} = 0.000001$.