Math 172 | Exam 1 | Name: ____________
31 May 2016 | Reasonably Simplify | Point Values in $\boxed{\text{boxes}}$.

1. Integrate.

(a) $\boxed{10}$ $\int t(t+12)^{11}\,dt = \int (u-12)u^{11}\,du = \int (u^{12} - 12u^{11})\,du$

Let $u = t+12$

$u - 12 = t$

$du = dt$

$= \frac{u^{13}}{13} - u^{12} + C$

$= \frac{(t+12)^{13}}{13} - (t+12)^{12} + C$

$= 1 + \tan^2 \pi x$

(b) $\boxed{10}$ $\int \sec^4(\pi x)\tan^3(\pi x)\,dx = \int \sec^2 \pi x \tan^3 \pi x \sec^2 \pi x\,dx$

$u = \tan \pi x$

$du = \sec^2 \pi x \cdot \pi\,dx$

$= \frac{1}{\pi}\int (1+u^2)u^3\,du = \frac{1}{\pi}\int (u^3 + u^5)\,du$

$= \frac{1}{\pi}\left(\frac{\tan^4 \pi x}{4} + \frac{\tan^6 \pi x}{6}\right) + C$

$\sec^2(\pi x) - 1$

OR

$= \int \sec^3(\pi x)\tan^2(\pi x)\sec(\pi x)\tan(\pi x)\,dx$

$u = \sec \pi x \qquad du = \sec \pi x \tan \pi x\,dx$

$= \frac{1}{\pi}\int u^3(u^2 - 1)\,du = \frac{1}{\pi}\int (u^5 - u^3)\,du$

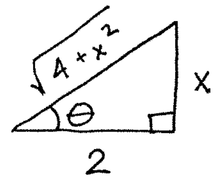
$= \frac{1}{\pi}\left(\frac{\sec^6 \pi x}{6} - \frac{\sec^4 \pi x}{4}\right) + C$

2. Integrate.

(a) $\boxed{10}$ $\displaystyle\int \frac{4}{(4+x^2)^2}\,dx$ [HINT: use an appropriate trig sub.]

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$



$4 + x^2 = 4 + 4\tan^2\theta = 4\sec^2\theta$

$$= \int \frac{4\cdot 2\sec^2\theta\, d\theta}{16\sec^4\theta} = \frac{1}{2}\int \cos^2\theta\, d\theta = \frac{1}{4}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{4}\left(\arctan\frac{x}{2} + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{4}\left(\arctan\frac{x}{2} + \frac{x}{\sqrt{4+x^2}}\cdot\frac{2}{\sqrt{4+x^2}}\right) + C = \frac{1}{4}\left(\arctan\frac{x}{2} + \frac{2x}{4+x^2}\right) + C$$

(b) $\boxed{10}$ $\displaystyle\int \frac{4x}{x^2-4x+8}\,dx = \int \frac{4x-8}{x^2-4x+8}\,dx + \int \frac{8}{(x-2)^2+2^2}\,dx$

$$= 2\ln\left(x^2-4x+8\right) + 4\arctan\left(\frac{x-2}{2}\right) + C$$

3. We are interested in evaluating $\int \ln(x^4-1)\,dx$.

(a) $\boxed{4}$ Use Integration by Parts to show $\int \ln(x^4-1)\,dx = x\ln(x^4-1) - \int \frac{4x^4}{x^4-1}\,dx$.

$u = \ln(x^4-1)$ $\qquad dv = dx$

$du = \frac{4x^3}{x^4-1}\,dx$ $\qquad v = x$

and the result follows

(b) $\boxed{10}$ Convert $\frac{4x^4}{x^4-1}$ into a proper fraction, then find its partial fraction decomposition.

$$\frac{4x^4}{x^4-1} = \frac{4x^4-4}{x^4-1} + \frac{4}{x^4-1} = 4 + \frac{4}{x^4-1}$$

$$\frac{4}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$4 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

Let $x=1$: $4 = A(4)$ so $A=1$ $\qquad \underline{x^3}$: $0 = A+B+C \Rightarrow C=0$

$x=-1$: $4 = B(-4)$ so $B=-1$ $\qquad \underline{x^0}$: $4 = A-B-D \Rightarrow D=-2$

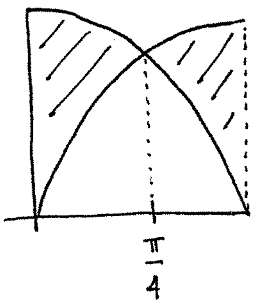
so $$\frac{4x^4}{x^4-1} = 4 + \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1}$$

(c) $\boxed{6}$ Using the decomposition you found above, finish the integration started in (a).

$$\int \ln(x^4-1)\,dx = x\ln(x^4-1) - \left[4x + \ln|x-1| - \ln|x+1| - 2\arctan x\right] + C$$

$$= x\ln(x^4-1) - 4x - \ln|x-1| + \ln|x+1| + 2\arctan x + C$$

4. $\boxed{10}$ Consider the region between the graphs of $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$. Carefully sketch the region and find the area.



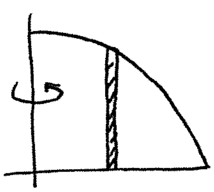
By symmetry, we consider only half

$$2\int_0^{\pi/4} (\cos x - \sin x)\,dx = 2\left[\sin x + \cos x\right]\Big|_0^{\pi/4}$$

$$= 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1\right)$$

$$= 2\left(\sqrt{2} - 1\right)$$

5. $\boxed{10}$ Consider the region bounded by the graph of $y = \cos x$ and the x-axis for $0 \leq x \leq \frac{\pi}{2}$. Carefully sketch the region. Find the volume of the solid generated by rotating the region about the y-axis.



$$2\pi\int_0^{\pi/2} x\cos x\,dx = 2\pi\left[x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right]$$

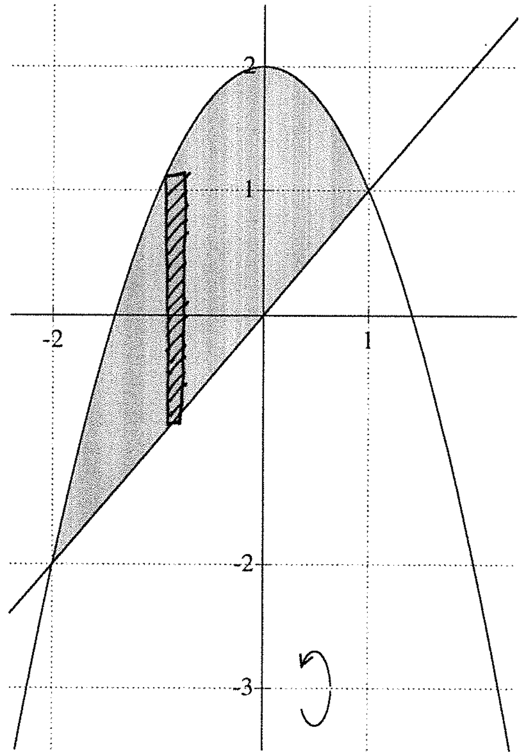
$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$= 2\pi\left[\frac{\pi}{2} + \cos x\Big|_0^{\pi/2}\right]$$

$$= 2\pi\left(\frac{\pi}{2} - 1\right) \text{ or } \pi(\pi - 2) \text{ or } \pi^2 - 2\pi$$

6. Consider the region bounded by the graphs of $y = 2 - x^2$ and $y = x$. The region is rotated about the line $y = -3$. See the figure below.



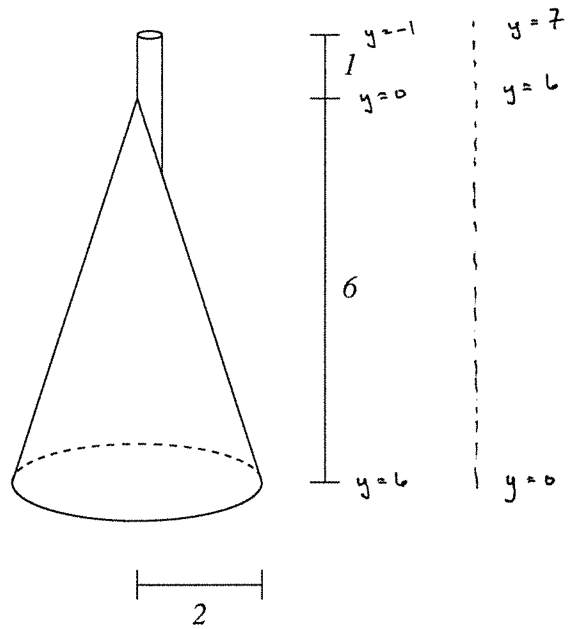
(a) $\boxed{2}$ Explain why using the Shell method would be inconvenient to compute the volume.

Shells would require two integrals.

(b) $\boxed{8}$ Express the volume of the solid using the Washer method. **Do not evaluate the integral.**

$$\pi \int_{-2}^{1} \left(\left[2 - x^2 + 3\right]^2 - \left[x + 3\right]^2 \right) dx \quad \text{or} \quad \pi \int_{-2}^{1} \left((5 - x^2)^2 - (x+3)^2 \right) dx$$

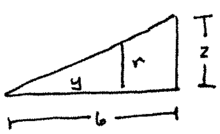
7. A conical tank on the moon is filled with liquid oxygen of density ρ. The gravitational constant on the moon is g. The tank is 6 m tall and has radius 2 m at the base. There is a spout protruding 1 m above the top of the cone. The cone is oriented as shown in the figure below.



(a) $\boxed{8}$ If the tank is full, express the work required to empty the tank through the spout as an integral. **Do not evaluate the integral.**

y = 0 at pointy/tip

distance $= (y - (-1)) = y + 1$



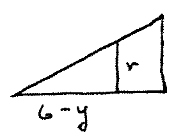
$\frac{r}{y} = \frac{2}{6}$ so $r = \frac{y}{3}$

force $= \frac{\pi}{9} y^2 \rho g$

$$\frac{\pi \rho g}{9} \int_0^6 y^2 (y+1)\, dy$$

y = 0 at the bottom

distance $= 7 - y$



$\frac{r}{6-y} = \frac{2}{6}$ so $r = \frac{1}{3}(6-y)$

force $= \frac{\pi}{9} (6-y)^2 \rho g$

$$\frac{\pi \rho g}{9} \int_0^6 (6-y)^2 (7-y)\, dy$$

(b) $\boxed{2}$ If the tank is 'half' full, i.e. the surface of the liquid oxygen is 3 m from the base, express the work required to empty the tank through the spout as an integral. **Do not evaluate the integral.** Do not reinvent the wheel, just make the needed change(s) to the integral above.

$$\frac{\pi \rho g}{9} \int_3^6 y^2 (y+1)\, dy$$

$$\frac{\pi \rho g}{9} \int_0^3 (6-y)^2 (7-y)\, dy$$