1. Integrate.

(a) \[ 10 \int t(t + 12)^{11} \, dt = \int (u - 12)^{12} \, du = \int (u^{13} - 12u^{12}) \, du \]

Let \( u = t + 12 \)
\[ u - 12 = t \]
\[ du = dt \]
\[ = \frac{u^{13}}{13} - u^{12} + C \]
\[ = \frac{(t + 12)^{13}}{13} - (t + 12)^{12} + C \]

(b) \[ 10 \int \sec^4(\pi x) \tan^3(\pi x) \, dx = \int \sec^2 \pi x \, t^{2} \sec^2 \pi x \, dx \]

Let \( u = t \pi x \)
\[ du = \sec^2 \pi x \, \pi dx \]
\[ = \frac{1}{\pi} \int (1 + u^2)u^3 \, du = \frac{1}{\pi} \int (u^5 + u^3) \, du \]
\[ = \frac{1}{\pi} \left( \frac{t \pi x^4}{4} + \frac{t \pi x^6}{6} \right) + C \]
\[ = \sec^2(\pi x) - 1 \]

or

\[ = \int \sec^3(\pi x) \, t^{2} \sec^2(\pi x) \, \sec(\pi x) \, \tan(\pi x) \, dx \]

Let \( u = \sec \pi x \)
\[ du = \sec \pi x \, \tan \pi x \, dx \]
\[ = \frac{1}{\pi} \int u^3 (u^2 - 1) \, du = \frac{1}{\pi} \int (u^5 - u^3) \, du \]
\[ = \frac{1}{\pi} \left( \frac{\sec^6(\pi x)}{6} - \frac{\sec^4(\pi x)}{4} \right) + C \]
2. Integrate.

(a) \[ 10 \int \frac{4}{(4 + x^2)^2} \, dx \]

[HINT: use an appropriate trig sub.]

Let \( x = 2 \tan \theta \)

\[ dx = 2 \sec^2 \theta \, d\theta \]

\[ 4 + x^2 = 4 + 4 \tan^2 \theta = 4 \sec^2 \theta \]

\[ = \frac{1}{4} \int \frac{4 \cdot 2 \sec^2 \theta \, d\theta}{\ln \sec^2 \theta} = \frac{1}{2} \int \cos^2 \theta \, d\theta = \frac{1}{4} \int (1 + \cos 2\theta) \, d\theta \]

\[ = \frac{1}{4} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{4} \left( \arctan \frac{x}{2} + \sin \theta \cos \theta \right) + C \]

\[ = \frac{1}{4} \left( \arctan \frac{x}{2} + \frac{x}{\sqrt{4 + x^2}} \cdot \frac{2}{\sqrt{4 + x^2}} \right) + C = \frac{1}{4} \left( \arctan \frac{x}{2} + \frac{2x}{4 + x^2} \right) + C \]

(b) \[ 10 \int \frac{4x}{x^2 - 4x + 8} \, dx \]

\[ = \int \frac{4x - 8}{x^2 - 4x + 8} \, dx + \int \frac{8}{(x-2)^2 + 2^2} \, dx \]

\[ = 2 \ln (x^2 - 4x + 8) + 4 \arctan \left( \frac{x-2}{2} \right) + C \]
3. We are interested in evaluating \( \int \ln(x^4 - 1) \, dx \).

(a) \[ 4 \] Use Integration by Parts to show \( \int \ln(x^4 - 1) \, dx = x \ln(x^4 - 1) - \int \frac{4x^4}{x^4 - 1} \, dx \).

\[
\begin{align*}
    u &= \ln(x^4 - 1) & dv &= dx \\
    du &= \frac{4x^3}{x^4 - 1} \, dx & v &= x
\end{align*}
\]

and the result follows.

(b) \[ 10 \] Convert \( \frac{4x^4}{x^4 - 1} \) into a proper fraction, then find its partial fraction decomposition.

\[
\frac{4x^4}{x^4 - 1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C + D}{x^2 + 1}
\]

\[ 4 = A(x+1)(x^2+1) + BS(x-1)(x^2+1) + (C + D)(x-1)(x+1) \]

\[
\begin{align*}
    \text{Let } x = 1: & \quad 4 = A(4) \quad \text{so} \quad A = 1 \\
    \text{Let } x = -1: & \quad 4 = B(-4) \quad \text{so} \quad B = -1
\end{align*}
\]

\[
\begin{align*}
    \frac{4}{x^4 - 1} &= \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2 + 1}
\end{align*}
\]

(c) \[ 6 \] Using the decomposition you found above, finish the integration started in (a).

\[
\int \ln(x^4 - 1) \, dx = x \ln(x^4 - 1) - \left[ 4x + \ln|x-1| - \ln|x+1| + 2 \arctan(x) \right] + C
\]

\[
= x \ln(x^4 - 1) - 4x - \ln|x-1| + \ln|x+1| + 2 \arctan(x) + C
\]
4. Consider the region between the graphs of $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$. Carefully sketch the region and find the area.

\[
\begin{align*}
\text{By symmetry, we consider only half:} & \\
2 \int_{0}^{\frac{\pi}{4}} (\cos x - \sin x) \, dx &= 2 \left[ \sin x + \cos x \right]_{0}^{\frac{\pi}{4}} \\
&= 2 \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 \right) \\
&= 2 \left( \sqrt{2} - 1 \right)
\end{align*}
\]

5. Consider the region bounded by the graph of $y = \cos x$ and the $x$-axis for $0 \leq x \leq \frac{\pi}{2}$. Carefully sketch the region. Find the volume of the solid generated by rotating the region about the $y$-axis.

\[
\begin{align*}
2\pi \int_{0}^{\frac{\pi}{2}} x \cos x \, dx &= 2\pi \left[ x \sin x \right]_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} \sin x \, dx \\
&= 2\pi \left( \frac{\pi}{2} + \cos x \right)_{0}^{\frac{\pi}{2}} \\
&= 2\pi \left( \frac{\pi}{2} - 1 \right)
\end{align*}
\]
6. Consider the region bounded by the graphs of \( y = 2 - x^2 \) and \( y = x \). The region is rotated about the line \( y = -3 \). See the figure below.

(a) \[\text{Shells would require two integrals.}\]

(b) Express the volume of the solid using the Washer method. \textbf{Do not evaluate the integral.}

\[
\pi \int_{-2}^{1} \left( \left[ 2 - x^2 + 3 \right]^2 - \left[ x + 3 \right]^2 \right) \, dx

\pi \int_{-2}^{1} \left( \left[ \frac{5 - x}{2} \right]^2 - \left[ x + 3 \right]^2 \right) \, dx
\]
7. A conical tank on the moon is filled with liquid oxygen of density $\rho$. The gravitational constant on the moon is $g$. The tank is 6 m tall and has radius 2 m at the base. There is a spout protruding 1 m above the top of the cone. The cone is oriented as shown in the figure below.

(a) If the tank is full, express the work required to empty the tank through the spout as an integral. **Do not evaluate the integral.**

\[
\int_0^6 \frac{\pi \rho g}{9} y^2 (y + 1) \, dy
\]

(b) If the tank is 'half' full, i.e. the surface of the liquid oxygen is 3 m from the base, express the work required to empty the tank through the spout as an integral. **Do not evaluate the integral.** Do not reinvent the wheel, just make the needed change(s) to the integral above.

\[
\int_0^3 \frac{\pi \rho g}{9} (6-y)^2 (7-y) \, dy
\]

\[
\int_0^3 \frac{\pi \rho g}{9} (6-y)^2 (7-y) \, dy
\]