

1. Integrate.

$$(a) \boxed{2} \int_0^{\pi/4} \tan x \sec^2 x \, dx = \int_u^1 du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$\frac{x}{0} \mapsto \frac{u}{0}$$

$$\frac{\pi}{4} \quad 1$$

$$(b) \boxed{4} \int_0^{\sqrt{3}} \frac{1+x}{1+x^2} \, dx = \int_0^{\sqrt{3}} \left(\frac{1}{1+x^2} + \frac{x}{1+x^2} \right) \, dx$$

$$= \arctan x \Big|_0^{\sqrt{3}} + \frac{1}{2} \ln(1+x^2) \Big|_0^{\sqrt{3}}$$

$$= \arctan(\sqrt{3}) - \arctan(0) + \frac{1}{2} \ln(1+3) - \frac{1}{2} \ln(1)$$

$$= \frac{\pi}{3} + \frac{1}{2} \ln 4 = \frac{\pi}{3} + \ln 2$$

2. Use the substitution $x = \sin^2 t$ to integrate the following¹.

$$\int \frac{dx}{\sqrt{x(1-x)}}$$

$$x = \sin^2 t$$

$$dx = 2 \sin t \cos t \, dt$$

$$= \int \frac{2 \sin t \cos t \, dt}{\sqrt{\sin^2 t (1-\sin^2 t)}} = \int 2 \, dt = 2t + C$$

$$= 2 \arcsin \sqrt{x} + C$$

so

$$\sqrt{x} = \sin t$$

$$t = \arcsin \sqrt{x}$$

¹This problem came from an old Soviet book of calculus problems I stumbled upon a few days ago.

3. On the Chapter 5 Thing we showed that for $a > 0$, $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$.

(a) [4] For $a > 0$, find a similar general form for $\int \frac{du}{\sqrt{a^2 - u^2}}$.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \frac{1}{a} \int \frac{du}{\sqrt{1 - \left(\frac{u}{a}\right)^2}} = \int \frac{dt}{\sqrt{1 - t^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\text{let } t = \frac{u}{a}$$

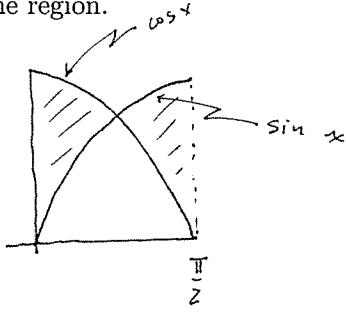
$$dt = \frac{1}{a} du$$

(b) [2] Using what you found above, integrate

$$\frac{1}{2} \int \frac{2 dx}{\sqrt{9 - 4x^2}}, \quad u = 2x, \quad du = 2 dx$$

$$= \frac{1}{2} \arcsin\left(\frac{2x}{3}\right) + C$$

4. [4] Sketch the region between the graphs of $y = \cos x$ and $y = \sin x$ for $x \in [0, \pi/2]$. Find the area of the region.



By symmetry we consider only the left half.

$$2 \int_0^{\pi/4} (\cos x - \sin x) dx = 2 \left[\sin x + \cos x \right] \Big|_0^{\pi/4}$$

$$= 2 [\sqrt{2} - 1]$$