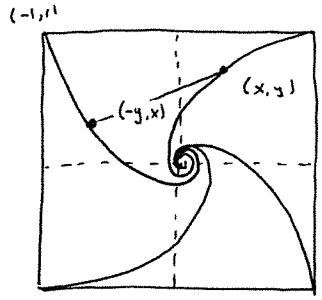
Dogs!



A) On the curve $r_1(\theta)$ starting at $(1,1)$ & spiralling in we know that at a generic (x,y), since the dog is always pointing toward the dog starting at $(-1,1)$, it is headed toward $(-y,x)$. That allows us to compute the slope as

$$\frac{dy}{dx} = \frac{x-y}{-y-x} = \frac{-y+x}{-x-y} = \frac{-r_1\sin\theta + r_1\cos\theta}{-r_1\cos\theta - r_1\sin\theta}. \quad (1)$$

We know the slope on a generic polar curve is given by $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$. (2)

It is clear by inspection that (1) & (2) coincide if $r' = -r$. (3)

The differential equation (3) has a general solution of $r = Ce^{-\theta}$.

Substituting the initial data gives,

$$r_1(\theta) = \sqrt{2}\,e^{-\theta + \pi/4}, \quad r_2(\theta) = \sqrt{2}\,e^{-\theta + \frac{3\pi}{4}}, \quad r_3(\theta) = \sqrt{2}\,e^{5\pi/4 - \theta}, \text{ and } r_4(\theta) = \sqrt{2}\,e^{7\pi/4 - \theta}$$

B) To find the arc length we note, by (3) $r' = -r$ so $ds = \sqrt{r^2 + (r')^2}\,d\theta = \sqrt{r^2 + r^2}\,d\theta$
$= \sqrt{2}\,r\,d\theta$.

$$s = \int_{\pi/4}^{\infty} \sqrt{2}\cdot\sqrt{2}\,e^{\pi/4 - \theta}\,d\theta = -2e^{\pi/4 - \theta}\Big|_{\pi/4}^{\infty} = 2$$

(Curiously, the distance they started apart.)

C. 5 seconds

D. since $\pi/4 \le \theta < \infty$ in 5 seconds, they get pretty dizzy.

It is worth noting that after one revolution each has covered a distance of $2 - \frac{2}{e^{2\pi}} \approx 1.996$, i.e. within 4 cm of the total distance.