Math 182 | Exam 3 | Name: _____

11 April 2017 | Provide appropriate justification. | Point Values in $\boxed{\text{boxes}}$.

1. $\boxed{8}$ Please circle **True** or **False**, as appropriate.

   (a) $\boxed{\mathbf{T}}$ / **F** : If $a_n \to 2$ as $n \to \infty$, the sequence $\{a_n\}$ converges.

   (b) **T** / $\boxed{\mathbf{F}}$: If the sequence $\{b_n\}$ is bounded, the sequence $\{b_n\}$ converges. $\{0, 1, 0, 1, 0, 1, \cdots\}$ is bounded & diverges

   (c) **T** / $\boxed{\mathbf{F}}$: If $a_n \to 0$ as $n \to \infty$, the series $\sum a_n$ converges.

   (d) **T** / $\boxed{\mathbf{F}}$: If $\sum a_n(x-3)^n$ has a radius of convergence $0 < R < \infty$, then for each $x \in \mathbb{R}$ $\sum a_n(x-3)^n$ either converges absolutely or diverges. May converge conditionally at end points $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges conditionally for $x=-1$

2. $\boxed{6}$ Find the sum of the following series.

   (a) $3 - \frac{6}{5} + \frac{12}{25} - \frac{24}{125} + \frac{48}{625} - \cdots = \frac{3}{1-(-2/5)} = \frac{3}{7/5} = \frac{15}{7}$

   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

3. $\boxed{6}$ Use the Maclaurin series for e^x to answer the following.

   (a) Approximate e^{-1} with $T_4(-1)$, i.e. the 4th Taylor polynomial - equivalently the 4th partial sum. Do not simplify, just write out the terms.

   $T_4(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}$

   (b) Yesterday we developed some reasonably complicated error estimates for Taylor series. Since the series above satisfies the Alternating Series Test, we can use a simplier, and stronger, error estimate. What is the error bound implied by the Alternating Series Test.

   $|e^{-1} - T_4(-1)| < \frac{1}{5!}$

Choose two of the three series to work with. Do not do all three!

4. $\boxed{20}$ Use appropriate tests to show each converges or diverges. Formal arguments are expected.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(2n+1)}{n^2+7}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n+3}{n^3-2}$

A. Since $\dfrac{1}{n\sqrt{\ln n}} > \dfrac{1}{(n+1)\sqrt{\ln(n+1)}} > 0$ for all $n \geq 2$ the C.C.T. applies.

we consider $\displaystyle\sum_{k=4}^{\infty} 2^k \cdot \frac{1}{2^k\sqrt{\ln 2^k}} = \sum_{k=4}^{\infty} \frac{1}{\sqrt{\ln 2}\cdot\sqrt{k}}$ which is a divergent p-series $(p = \frac{1}{2} \leq 1)$.

By the Cauchy Condensation Test $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ also diverges.

<u>Note</u>: Integral Test also applies

B. $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(2n+1)}{n^2+7}$ <u>converges absolutely</u> by comparison to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is a convergent p-series $(p=2>1)$

since $0 < \left| \dfrac{\sin(2n+1)}{n^2+7} \right| < \dfrac{1}{n^2}$.

C. Initially we note that $\dfrac{n+3}{n^3-2} > 0$ & $\dfrac{1}{n^2} > 0$ for $n \geq 2$.

For large n, $\dfrac{n+3}{n^3-2}$ is similar to $\dfrac{1}{n^2}$, specifically,

$\displaystyle\lim_{n\to\infty} \frac{n+3}{n^3-2} \Big/ \frac{1}{n^2} = 1$. Since the limit is positive

and finite, the series behave the same. We know

$\sum \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$, so by

the Limit Comparison Test $\sum \frac{n+3}{n^3-2}$ also converges.

5. $\boxed{15}$ Determine if the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$$

Initially we test for absolute convergence.

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}$$ which diverges by comparison to the Harmonic series

since $0 < \frac{1}{n} < \frac{\ln n}{n}$.

We conclude that $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ does not converge absolutely. (1)

We can apply the Alternating Series Test to test for convergence.

Since $\frac{\ln n}{n} > 0$ for $n \geq 2$ & $\frac{\ln n}{n} \xrightarrow[n\to\infty]{} 0$

we only need $\frac{\ln n}{n}$ to be decreasing. Consider the continuous version, $f(x) = \frac{\ln x}{x}$.

$f'(x) = \frac{1 - \ln x}{x^2}$ which is negative for $x > e$. Since $\frac{\ln x}{x}$ is decreasing, $\frac{\ln n}{n}$ is

decreasing and the Alternating Series Test implies $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges. (2)

By (1) and (2) we conclude $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ <u>converges conditionally</u>.

6. $\boxed{5}$ Find the radius of convergence for the following series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^2}{(n^3+7)2^{2n+3}}$$

typo! Obviously should be $(x-3)^n$. As written, converges for all x since $(x-3)^2$ is just a constant.

We apply the Root test (to the series intended).

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(-1)^n (x-3)^n}{(n^3+7)2^{2n+3}}\right|} = \frac{|x-3|}{4} \sqrt[n]{\frac{1}{8(n^3+7)}} \xrightarrow[n\to\infty]{} \frac{|x-3|}{4} < 1$$

so $|x-3| < 4$,

i.e. $R = 4$

7. $\boxed{10}$ Find a series representation for $f(x) = \dfrac{4}{2+x}$ centered at $x = 2$ and find the interval of convergence.

$$f(x) = \frac{4}{2+x} = \frac{4}{2+(x-2)+2} = \frac{4}{4+(x-2)} = \frac{1}{1+\left(\frac{x-2}{4}\right)} = \frac{1}{1-\left(\frac{x-2}{-4}\right)}$$

$$= \sum_{n=0}^{\infty} \left(\frac{x-2}{-4}\right)^n \quad \text{for} \quad \left|\frac{x-2}{-4}\right| < 1$$

i.e. $|x-2| < 4$

OR $x \in (-2, 6)$

8. $\boxed{5}$ Find a series representation for $F(x) = \int_0^x e^{x^2}\,dx$. ← Yet another typo, should be $\int_0^x e^{t^2}\,dt$

$$e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \quad \text{so} \quad \int_0^x \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\,dt$$

$$= A + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)n!}\Bigg|_0^x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$$