1. [8] Please circle True or False, as appropriate.
   
   (a) [T] F : If \( a_n \to 2 \) as \( n \to \infty \), the sequence \( \{a_n\} \) converges.
   
   (b) T [F] : If the sequence \( \{b_n\} \) is bounded, the sequence \( \{b_n\} \) converges. \( \{0, 1, 0, 1, 0, \ldots\} \) is bounded & diverges.
   
   (c) T [F] : If \( a_n \to 0 \) as \( n \to \infty \), the series \( \sum a_n \) converges.
   
   (d) T [F] : If \( \sum a_n(x - 3)^n \) has a radius of convergence \( 0 < R < \infty \), then for each \( x \in \mathbb{R} \) \( \sum a_n(x - 3)^n \) either converges absolutely or diverges. My converge conditionally at each points \( \sum_{n=1}^{\infty} \frac{x^n}{n} \) converges conditionally for \( x = 1 \).

2. [6] Find the sum of the following series.
   
   (a) \( 3 - \frac{6}{5} + \frac{12}{25} - \frac{24}{125} + \frac{48}{625} - \cdots = \frac{3}{1 - (-\frac{3}{5})} = \frac{15}{7} \)
   
   (b) \( \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \cos \left( \frac{\pi}{3} \right) \times \frac{1}{2} \)

3. [6] Use the Maclaurin series for \( e^x \) to answer the following.
   
   (a) Approximate \( e^{-1} \) with \( T_4(-1) \), i.e. the 4th Taylor polynomial - equivalently the 4th partial sum. Do not simplify, just write out the terms.
   
   \[ T_4(-1) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \]

   (b) Yesterday we developed some reasonably complicated error estimates for Taylor series. Since the series above satisfies the Alternating Series Test, we can use a simpler, and stronger, error estimate. What is the error bound implied by the Alternating Series Test.
   
   \[ |e^{-1} - T_4(-1)| < \frac{1}{5!} \]
Choose two of the three series to work with. Do not do all three!

4. Use appropriate tests to show each converges or diverges. Formal arguments are expected.

(a) \[ \sum_{n=2}^{\infty} \frac{1}{n^3 \ln n} \]

(b) \[ \sum_{n=2}^{\infty} \frac{\sin (2n+1)}{n^2 + 7} \]

(c) \[ \sum_{n=2}^{\infty} \frac{n + 3}{n^3 - 2} \]

A. Since \[ \frac{1}{n^3 \ln n} > \frac{1}{(kn)^3 \ln (kn)} \] for all \( n \geq 2 \), the C.C.T. applies.

We consider

\[ \sum_{k=1}^{\infty} \frac{2^k \cdot 1}{2^k \sqrt{k^2 - 1k}} = \sum_{k=1}^{\infty} \frac{1}{k \sqrt{k^2 - 1k}} \]

which is a divergent p-series \( (p = \frac{1}{2} < 1) \).

By the Cauchy Condensation Test, \[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \] also diverges.

Note: Integral Test also applies.

B. \[ \sum_{n=2}^{\infty} \frac{\sin (2n+1)}{n^2 + 7} \] converges absolutely by comparison to \[ \sum_{n=2}^{\infty} \frac{1}{n^2} \] which is a convergent p-series \( (p=2>1) \).

Since \( 0 < \left| \frac{\sin (2n+1)}{n^2 + 7} \right| < \frac{1}{n^2} \).

C. Initially we note that \( \frac{n^3}{n^3 - 2} > 0 \) and \( \frac{1}{n^2} > 0 \) for \( n \geq 2 \).

For large \( n \), \[ \frac{n^3}{n^3 - 2} \] is similar to \( \frac{1}{n^2} \), specifically,

\[ \lim_{n \to \infty} \frac{n^3}{n^3 - 2} / \frac{1}{n^2} = 1 \]. Since the limit is positive and finite, the series behave the same. We know \[ \sum \frac{1}{n^2} \] is a convergent p-series \( (p=2>1) \), so by

the Limit Comparison Test, \[ \sum \frac{n^3}{n^3 - 2} \] also converges.
5. Determine if the following series converges absolutely, converges conditionally, or diverges.

\[
\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}
\]

Initially we test for absolute convergence.

\[
\sum_{n=2}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}
\]

which diverges by comparison to the harmonic series

since \( 0 < \frac{1}{n} < \frac{\ln n}{n} \).

We conclude that \( \sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n} \) does not converge absolutely. (1)

We can apply the Alternating Series Test to test for convergence.

Since \( \frac{\ln n}{n} > 0 \) for \( n \geq 2 \) and \( \frac{\ln n}{n} \) \( \xrightarrow[n \to \infty]{} 0 \)

we only need \( \frac{\ln n}{n} \) to be decreasing. Consider the continuous version, \( f(x) = \frac{\ln x}{x} \).

\( f'(x) = \frac{1 - \ln x}{x^2} \) which is negative for \( x > e \). Since \( \frac{\ln x}{x} \) is decreasing, \( \frac{\ln n}{n} \) is decreasing and the Alternating Series Test implies \( \sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n} \) converges. (2)

By (1) and (2) we conclude \( \sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n} \) converges conditionally.
6. Find the radius of convergence for the following series.

\[ \sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^2}{(n^3+7)^{2n+3}} \]  

As written, converges for all \( x \) since \((x-3)^2\) is just a constant.

We apply the Root test (to the series intended).

\[ n \left( |x-3| \right) = \sqrt[n]{\frac{(-1)^n(x-3)^{2n}}{(n^3+7)^{2n+3}}} = \frac{|x-3|}{\sqrt[n]{8(n^3+7)}} \xrightarrow{n \to \infty} \frac{|x-3|}{4} < 1 \]

\[ |x-3| < 4, \]

i.e. \( R = 4 \)

7. Find a series representation for \( f(x) = \frac{4}{2+x} \) centered at \( x = 2 \) and find the interval of convergence.

\[ f(x) = \frac{4}{2+x} = \frac{4}{2+(x-2)+2} = \frac{4}{4+(x-2)} = \frac{1}{1+\left(\frac{x-2}{4}\right)} = \frac{1}{\frac{x-2}{4}} \]

\[ = \sum_{n=0}^{\infty} \left(\frac{x-2}{4}\right)^n \]

\[ f(x) \quad \left| \frac{x-2}{4} \right| < 1 \]

i.e. \( |x-2| < 4 \)

or \( x \in (-2, 6) \)

8. Find a series representation for \( F(x) = \int_0^x e^{t^2} \, dt \).

\[ e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \quad \Rightarrow \int_0^x e^{t^2} \, dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} \]