

Math 172
2 May 2017

Final Exam
Show Appropriate Work

Name: _____
Point Values in boxes.

1. Integrate.

$$(a) \boxed{10} \int_{-\infty}^0 xe^{\pi x} dx = \frac{xe^{\pi x}}{\pi} \Big|_{-\infty}^0 - \int_{-\infty}^0 \frac{1}{\pi} e^{\pi x} dx = \frac{xe^{\pi x}}{\pi} - \frac{e^{\pi x}}{\pi^2} \Big|_{-\infty}^0$$

$$u = x \quad dv = e^{\pi x} dx$$

$$du = dx \quad v = \frac{1}{\pi} e^{\pi x}$$

$$= \lim_{R \rightarrow -\infty} \left[0 - \frac{1}{\pi^2} - \frac{Re^{\pi R}}{\pi^2} + \frac{e^{\pi R}}{\pi^2} \right]$$

$$= -\frac{1}{\pi^2} + \lim_{R \rightarrow -\infty} \frac{-R}{\pi^2 e^{-\pi R}} \xrightarrow{\infty \text{ form}}$$

$$\stackrel{L'H}{=} -\frac{1}{\pi^2} + \lim_{R \rightarrow -\infty} \frac{-1}{-\pi^3 e^{-\pi R}} = -\frac{1}{\pi^2}$$

$$(b) \boxed{10} \int \frac{17}{(2t+1)(t^2+4)} dt = \int \left(\frac{4}{2t+1} - \frac{2t}{t^2+4} + \frac{1}{t^2+4} \right) dt$$

$$\frac{17}{(2t+1)(t^2+4)} = \frac{A}{2t+1} + \frac{Bt+C}{t^2+4}$$

$$17 = A(t^2+4) + (Bt+C)(2t+1)$$

$$\text{Let } t = -\frac{1}{2} : 17 = A \left(\frac{17}{4} \right) \text{ so } A = 4$$

Eq. Ceff

$$t^2 : 0 = A + 2B \text{ so } B = -2$$

$$t^0 : 17 = 4A + C \text{ so } C = 1$$

2. [5] Integrate $\int_0^x e^{x^3} dx$.

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \quad \text{so} \quad \int_0^x e^{x^3} dx = \sum_{n=0}^{\infty} \left[\frac{x^{3n+1}}{(3n+1)n!} \right]_0^x = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}$$

3. Let $f(x) = \sum_{n=0}^{\infty} \frac{n^2(x+7)^{2n}}{(n+1)4^{2n+1}}$.

(a) [6] Find the radius of convergence of the series.

We will use the Root Test.

$$\sqrt[n]{\left| \frac{n^2 (x+7)^{2n}}{(n+1)4^{2n+1}} \right|} \xrightarrow{n \rightarrow \infty} \frac{|x+7|^2}{4^2} < 1, \text{ so } |x+7| < 4.$$

The radius of convergence is 4.

(b) [4] Evaluate the following derivatives, do not simplify the arithmetic.

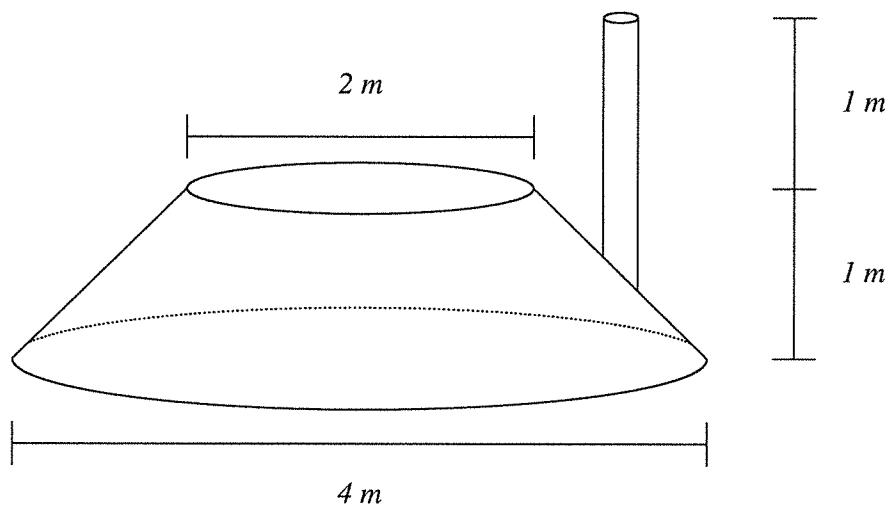
i. $f^{(2016)}(-7)$

The $(x+7)^{2016}$ term is the $n=1008$ term in the series, so

$$f^{(2016)}(-7) = \frac{(2016)! (1008)^2}{(1009) \cdot 4^{2017}}$$

ii. $f^{(2017)}(-7) = 0$

4. [10] A tank on the moon is filled with liquid oxygen of density ρ . The gravitational constant on the moon is g . The tank has the shape of a frustum of a cone (a cone with the top cut off) with dimensions as shown in the figure . There is a spout protruding 1 m above the top of the tank.



Find the work required to empty the full tank through the spout.

$$\begin{aligned}
 & \text{Diagram shows a frustum of a cone with a vertical axis. The top radius is } r = y, \text{ and the bottom radius is } r = 2y. \text{ The height of the frustum is } 1 \text{ m.} \\
 & \Delta W = f \cdot g \cdot \pi \cdot (\text{radius})^2 \cdot (\text{distance}) \Delta y \\
 & = f \cdot g \cdot \pi \cdot (y)^2 \cdot y \Delta y \\
 & \text{work} = fg\pi \int_1^2 y^3 dy = \left. \frac{fg\pi y^4}{4} \right|_1^2 = \frac{15fg\pi}{4}
 \end{aligned}$$

5. 4 Find parametric equations, $x(t)$ and $y(t)$, for the line segment from $(5, 4)$ to $(1, 9)$. Include an appropriate domain for t .

$$\begin{aligned}x(t) &= 5 - 4t \\y(t) &= 4 + 5t \quad \text{for } 0 \leq t \leq 1\end{aligned}$$

6. Consider the parametric curve given by $c(t) = (27 \cdot 4e^t, 27 \cdot 3e^{3t/2})$.

[HINT: for a constant c , $(27 \cdot c)^2 = 27^2 \cdot c^2$.]

$$\begin{aligned}x' &= 27 \cdot 4e^t \\y' &= 27 \cdot 3e^{3t/2}\end{aligned}$$

- (a) 3 Find the slope, $\frac{dy}{dx}$, of the curve at $t = 2$

$$\left. \frac{dy}{dx} \right|_{t=2} = \frac{27 \cdot 3e^3}{27 \cdot 4e^2} = \frac{3e}{4}$$

- (b) 3 Find the speed, $\frac{ds}{dt}$, of a particle traveling along the curve at $t = 0$.

$$\left. \frac{ds}{dt} \right|_{t=0} = \sqrt{(27 \cdot 4)^2 + (27 \cdot 3)^2} = 27 \sqrt{4^2 + 3^2} = 27 \cdot 5 = 135$$

- (c) 10 Find the length of the curve for $t \in [0, \ln(\frac{20}{9})]$. Simplify your solution.

$$(x')^2 + (y')^2 = 27^2 \cdot 4^2 e^{2t} + 27^2 \cdot 3^2 e^{3t} = 27^2 e^{2t} (4^2 + 3^2 e^t), \text{ so}$$

$$ds = 27e^t \sqrt{16 + 9e^t} dt$$

$$s = 27 \int_0^{\ln(\frac{20}{9})} e^t \sqrt{16 + 9e^t} dt = 3 \int_{25}^{36} u^{\frac{1}{2}} du = 2u^{\frac{3}{2}} \Big|_{25}^{36} = 2 \left(36^{\frac{3}{2}} - 25^{\frac{3}{2}} \right)$$

$$u = 16 + 9e^t$$

$$du = 9e^t dt$$

$$0 \longmapsto 25$$

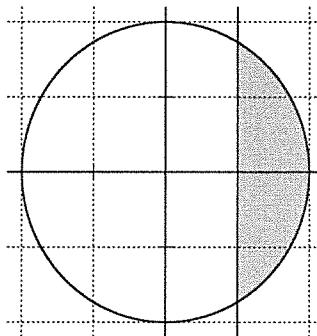
$$\ln(\frac{20}{9}) \longmapsto 36$$

$$= 2 (216 - 125)$$

$$= 2 (91)$$

$$= 182 \leftarrow \text{Huzzah!}$$

7. Consider the region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$, the shaded region in the figure.



8. [10] Express the area of the region as an integral in rectangular coordinates. Use an appropriate trigonometric substitution to evaluate the integral.

$$2 \int_{-1}^2 \sqrt{4 - x^2} dx = 2 \int_{\pi/6}^{\pi/2} 2 \cos \theta \cdot 2 \cos \theta d\theta = 4 \int_{\pi/6}^{\pi/2} 2 \cos^2 \theta d\theta = 4 \int_{\pi/6}^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$\text{Let } x = 2 \sin \theta$$

$$dx = 2 \cos \theta d\theta$$

$$= 4 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\pi/6}^{\pi/2}$$

$$1 \mapsto \frac{\pi}{6}$$

$$= 4 \left(\frac{\pi}{2} - \frac{\pi}{6} + \left(0 - \frac{\sqrt{3}}{4} \right) \right)$$

$$2 \mapsto \frac{\pi}{2}$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

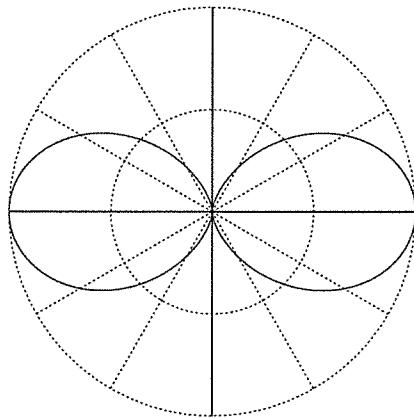
9. [10] Express the area of the region as an integral in polar coordinates. Evaluate the integral.

$$\begin{aligned} x^2 + y^2 &= 4 & \text{so} & \quad r = 2 \\ x &= 1 & \text{so} & \quad r = \sec \theta \end{aligned} \quad \left. \begin{array}{l} \text{intersect when } \theta = \frac{\pi}{3} \end{array} \right\}$$

By symmetry

$$2 \cdot \frac{1}{2} \int_0^{\pi/3} (4 - \sec^2 \theta) d\theta = 4\theta - \tan \theta \Big|_0^{\pi/3} = \frac{4\pi}{3} - \sqrt{3}$$

10. Consider the polar curve given by $r = \cos^2 \theta$.



11. [13] Find the length of the curve. Use symmetry and consider only the part given by $0 \leq \theta \leq \pi/2$.

$$r = \cos^2 \theta$$

$$r' = -2\cos \theta \sin \theta$$

$$\begin{aligned} r^2 + (r')^2 &= \cos^4 \theta + 4\cos^2 \theta \sin^2 \theta \\ &= \cos^2 \theta (\cos^2 \theta + 4\sin^2 \theta) \end{aligned}$$

$$(*) \quad ds = \cos \theta \sqrt{1+3\sin^2 \theta} d\theta$$

$$s = 4 \int_0^{\pi/2} \cos \theta \sqrt{1+3\sin^2 \theta} d\theta$$

$$\begin{aligned} \text{Let } \sqrt{3}\sin \theta &= \tan \varphi \\ \text{so } \sqrt{3}\cos \theta &= \sec^2 \varphi d\varphi \end{aligned}$$

$$\begin{aligned} \theta = 0 &\longrightarrow \varphi = 0 \\ \theta = \frac{\pi}{2} &\longrightarrow \varphi = \frac{\pi}{3} \end{aligned}$$

$$= \frac{4}{\sqrt{3}} \int_0^{\pi/3} \sec^3 \varphi d\varphi$$

$$= \frac{2}{\sqrt{3}} \left[\sec \varphi \tan \varphi + \ln |\sec \varphi + \tan \varphi| \right] \Big|_0^{\pi/3}$$

$$= \frac{2}{\sqrt{3}} \left[2\sqrt{3} + \ln(2+\sqrt{3}) \right]$$

12. [2] There is a good mathematical reason to only consider $0 \leq \theta \leq \pi/2$. What is the complication if we were to instead consider $0 \leq \theta \leq \pi$ or $0 \leq \theta \leq 2\pi$?

In (*) we should get $ds = |\cos \theta| \sqrt{1+3\sin^2 \theta} d\theta$. For $\theta \in [0, \frac{\pi}{2}]$, $\cos \theta \geq 0$, so $|\cos \theta| = \cos \theta$. For either $\theta \in [0, \pi]$ or $\theta \in [0, 2\pi]$ we would need multiple integrals to account for the absolute value.

Some trigonometric identities.

$$\cos^2 x = \frac{1}{2} (1 + \cos 2x) \quad \sin^2 x = \frac{1}{2} (1 - \cos 2x) \quad \sin 2x = 2 \sin x \cos x$$

Some integrals.

$$\begin{aligned}\int \frac{du}{a^2 + u^2} &= \frac{1}{a} \arctan\left(\frac{u}{a}\right) + c \\ \int \frac{du}{\sqrt{a^2 - u^2}} &= \arcsin\left(\frac{u}{a}\right) + c \\ \int \sec u \, du &= \ln |\sec u + \tan u| + c \\ \int \sec^3 u \, du &= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + c \\ \int \csc u \, du &= \ln |\csc u - \cot u| + c\end{aligned}$$

Some Maclaurin series.

Values of x where series converge is indicated in each case.

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &\text{all } x \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots &\text{all } x \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots &\text{all } x \\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &= 1 + x + x^2 + x^3 + \dots &|x| < 1 \\ \ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots &|x| < 1, x = 1 \\ \arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots &|x| \leq 1\end{aligned}$$