

Before you start writing on this, consider the use of scratch paper.

You will find the following identities useful.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

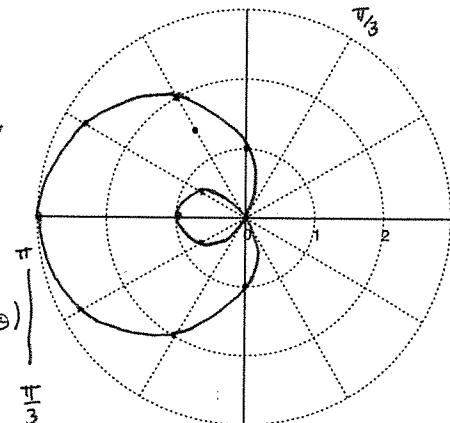
1. 4 Carefully sketch the graph of the limaçon¹ $r = 1 - 2\cos\theta$. Find the area between the inner and outer loops of the curve.

Inside outer loop

$$2 \cdot \frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} (1 - 2\cos\theta)^2 d\theta = \int_{\frac{\pi}{3}}^{\pi} (1 - 4\cos\theta + 4\cos^2\theta) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi} (3 - 4\cos\theta + 2\cos 2\theta) d\theta = (3\theta - 4\sin\theta + \sin 2\theta) \Big|_{\frac{\pi}{3}}^{\pi}$$

$$= 3\left(\pi - \frac{\pi}{3}\right) - 4\left(0 - \frac{\sqrt{3}}{2}\right) + \left(0 - \frac{\sqrt{3}}{2}\right) = 2\pi + \frac{3\sqrt{3}}{2}$$



Inside inner loop

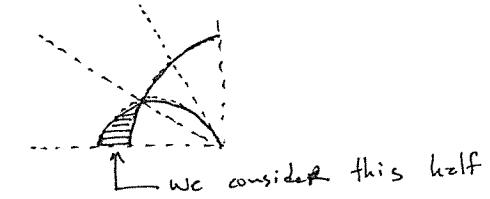
$$2 \cdot \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 - 2\cos\theta)^2 d\theta = (3\theta - 4\sin\theta + \sin 2\theta) \Big|_0^{\frac{\pi}{3}} = \pi - 4\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \pi - \frac{3\sqrt{3}}{2}$$

Subtracting gives

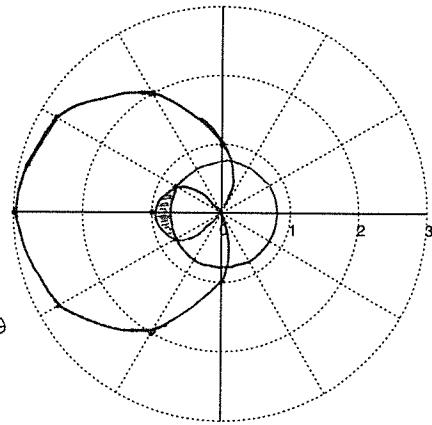
$$2\pi + \frac{3\sqrt{3}}{2} - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$$

¹The term derives from the French word limaçon, which means "snail."

2. [3] Carefully sketch the graphs of the limaçon $r = 1 - 2 \cos \theta$ and the circle $r = \sqrt{3} - 1 \approx 0.73$. Find the area inside the inner loop of the limaçon and outside the circle.



$$\begin{aligned}
 & 2 \cdot \frac{1}{2} \int_0^{\pi/6} \left[(1 - 2 \cos \theta)^2 - (\sqrt{3} - 1)^2 \right] d\theta \\
 &= \int_0^{\pi/6} (1 - 4 \cos \theta + 4 \cos^2 \theta - (3 - 2\sqrt{3} + 1)) d\theta \\
 &= \int_0^{\pi/6} (2\sqrt{3} - 1 - 4 \cos \theta + 2 \cos 2\theta) d\theta = (2\sqrt{3} - 1)\theta - 4 \sin \theta + \sin 2\theta \Big|_0^{\pi/6} \\
 &= (2\sqrt{3} - 1) \frac{\pi}{6} - 2 + \frac{\sqrt{3}}{2}
 \end{aligned}$$



3. [3] Carefully sketch the graph of the cardioid $r = 1 - \cos \theta$ and find then find the length of the curve.

$$\begin{aligned}
 r &= 1 - \cos \theta & r^2 &= 1 - 2 \cos \theta + \cos^2 \theta \\
 r' &= \sin \theta & (r')^2 &= \sin^2 \theta \\
 r^2 + (r')^2 &= \cancel{2-2\cos\theta} \quad 2 - 2 \cos \theta \\
 &= 2(1 - \cos \theta) & 1 - \cos \theta &= 2 \sin^2 \left(\frac{\theta}{2}\right) \\
 &= 4 \sin^2 \left(\frac{\theta}{2}\right)
 \end{aligned}$$

$$\int_0^{2\pi} 2 \sin \left(\frac{\theta}{2}\right) d\theta = -4 \cos \left(\frac{\theta}{2}\right) \Big|_0^{2\pi} = -4(-1 - 1) = 8$$

