Math 182 Thing **Name:** Key

Section: 6.1

16 Jan 2018

My hope is that you will finish this in class. Solutions will be posted later today. If you do not finish in class, please finish before class on Thursday and check your own work.

Definite integrals are signed areas between the graph of a function and the x-axis. For example,

$$\int_0^1 (x^2 - 1)\, dx = \frac{-2}{3}.$$

Since the function is negative on $x \in [0, 1]$ the definite integral is negative. On the other hand, when we want the area between two curves we always measure from the bottom curve to the top curve giving a positive height and hence area. In general, for area between the graphs of two functions, we compute

$$\int_a^b \text{top} - \text{bottom}.$$

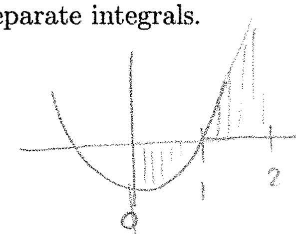
For example, the area between the x-axis and the graph of $y = x^2 - 1$ for $x \in [0, 1]$ we have

$$\int_0^1 \text{top} - \text{bottom}\, dx = \int_0^1 (0 - (x^2 - 1))\, dx = \int_0^1 (1 - x^2)\, dx = \frac{2}{3}.$$

We will explore this topic with the following examples.

1. Often the 'top' curve and the 'bottom' curve intersect and change. For example, to find the area between the graph of $y = x^2 - 1$ and the x-axis for $x \in [0, 2]$ we use two separate integrals.

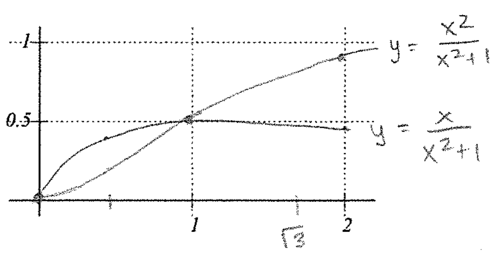
$$\int_0^1 (1 - x^2)\, dx + \int_1^2 (x^2 - 1)\, dx$$



Find this area.

$$A = \frac{2}{3} + \left(\frac{x^3}{3} - x \Big|_1^2 \right) = \frac{2}{3} + \left(\frac{8}{3} - 2 - \left(\frac{1}{3} - 1 \right) \right) = 2$$

2. Carefully sketch the graphs of $y = x^2/(x^2 + 1)$ and $y = x/(x^2 + 1)$ and then find the area between the graphs for $x \in [0, \sqrt{3}]$.



$$A = \int_0^1 \frac{x}{x^2+1} - \frac{x^2}{x^2+1}\, dx + \int_1^{\sqrt{3}} \frac{x^2}{x^2+1} - \frac{x}{x^2+1}\, dx$$

$$= \underbrace{\int_0^1 \frac{x}{x^2+1}\, dx}_{\substack{u = x^2+1 \\ du = 2x\,dx}} - \underbrace{\int_0^1 \frac{x^2}{x^2+1}\, dx}_{\pm 1 \text{ in numerator}} + \int_1^{\sqrt{3}} \frac{x^2}{x^2+1}\, dx - \underbrace{\int_1^{\sqrt{3}} \frac{x}{x^2+1}\, dx}_{u\text{-sub}}$$

$$= \int_1^2 \frac{1}{2u}\, du - \int_0^1 \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\, dx + \int_1^{\sqrt{3}} \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\, dx - \int_2^4 \frac{1}{2u}\, du$$

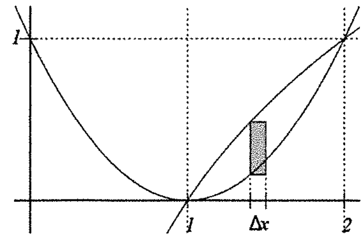
$$= \frac{1}{2} \ln u \Big|_1^2 - \left[(x - \arctan x \Big|_0^1 \right] + \left[(x - \arctan x) \Big|_1^{\sqrt{3}} \right] - \frac{1}{2} \ln u \Big|_2^4$$

$$= \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 - \left[1 - \pi/4 - 0 \right] + \left[\sqrt{3} - \pi/3 - (1 - \pi/4) \right] - \left(\frac{1}{2} \ln 4 - \frac{1}{2} \ln 2 \right)$$

$$= -2 + \pi/2 + \sqrt{3} - \pi/3$$

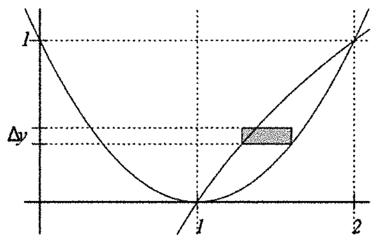
$y = \log_2 x$

3. Consider the region bounded by the graphs of $y = (x-1)^2$ and $\log_2 x$. Slicing this region up with respect to x leads to the following integral.



$$\int_1^2 (\log_2 x - (x-1)^2)\, dx$$

This is currently a problem for us since we don't know an antiderivative for $\log_2 x$ (we will return to this topic in Chapter 7). In this case, we want to consider slicing up the region with respect to y. Express the area as an integral with respect to y and then compute the area.



$y = (x-1)^2 \iff \pm\sqrt{y} + 1 = x$

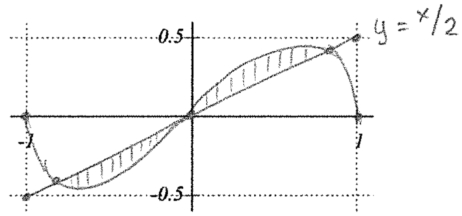
$y = \log_2 x \iff 2^y = x$

$A = \int_c^d \text{right} - \text{left}$

$$A = \int_0^1 \sqrt{y} + 1 - 2^y\, dy = \frac{2}{3} y^{3/2} + y - \frac{1}{\ln 2} 2^y \Big|_0^1$$

$$= \left(\frac{2}{3} + 1 - \frac{1}{\ln 2}\cdot 2\right) - \left(-\frac{1}{\ln 2}\cdot 1\right) = \frac{5}{3} - \frac{1}{\ln 2}$$

4. Carefully sketch the graphs of of $y = x/2$ and $y = x\sqrt{1-x^2}$ and then find the area bounded by the graphs.



intersect:

$\frac{x}{2} = x\sqrt{1-x^2}$

$\frac{x}{2} - x\sqrt{1-x^2} = 0$

$x\left(\frac{1}{2} - \sqrt{1-x^2}\right) = 0$

$x = 0$ or $\sqrt{1-x^2} = \frac{1}{2}$

$1 - x^2 = 1/4$

$\pm\sqrt{3}/2 = x$

$$A = \int_{-\sqrt{3}/2}^{0} x/2 - x\sqrt{1-x^2}\, dx + \int_0^{\sqrt{3}/2} x\sqrt{1-x^2} - x/2\, dx$$

$$= 2\int_0^{\sqrt{3}/2} x\sqrt{1-x^2} - x/2\, dx = 2\left[\int_0^{\sqrt{3}/2} x\sqrt{1-x^2}\, dx - \int_0^{\sqrt{3}/2} \frac{x}{2}\, dx\right]$$

$u = 1 - x^2$

$du = -2x\, dx$

$$= 2\left[-\frac{1}{2}\int_1^{1/4} \sqrt{u}\, du - \frac{1}{2}\int_0^{\sqrt{3}/2} x\, dx\right] = \frac{2}{3}u^{3/2}\Big|_{1/4}^{1} - \frac{x^2}{2}\Big|_0^{\sqrt{3}/2} = \frac{2}{3} - \frac{1}{12} - \frac{3}{8} = \frac{5}{24}$$