Math 172 Integration Review

- U-Substitution: Look for a function and its derivative to both be included in the integral (5.7)
- Integration by Parts - IBP: Look for a product of two functions in the integral (7.1)
- Trig Integrals: "Save" out a term - that becomes your du. That can help in establishing your u. Then use $\cos^2\theta + \sin^2\theta = 1$ or $\tan^2\theta + 1 = \sec^2\theta$ as appropriate to replace terms with u. Note: we use the half-angle identities when dealing with $\sin^2\theta$ or $\cos^2\theta$ (7.2)
- Trig Sub: Use the patterns (ex. $\sqrt{a^2 - x^2}$ implies you set $x = a\sin\theta$) to do your sub. Use a triangle to get back to x's for an indefinite integral. (7.3)
- Partial Fractions: Set up $\frac{A}{factor} + \frac{B}{factor} +$ etc. Set this equal to the rational expression in the integral. Eliminate fractions and then find the coefficients A, B, etc. Plug those coefficients back into your setup form and integrate. We often see $\ln x$, $\arctan x$ and u-sub when integrating. (7.5)

Evaluate the following integrals.

1. $\int xe^{2x}\,dx$ IBP
2. $\int \tan^3\theta\sec\theta\,d\theta$ trig identities & u-sub
3. $\int x\sqrt{x^2+1}\,dx$ u-sub
4. $\int \cos^2(3\theta)\,d\theta$ half angle identity
5. $\int \frac{4}{4-x^2}\,dx$ partial fractions or trig sub
6. $\int \frac{1}{\sqrt{x^2+4}}\,dx$ trig sub
7. $\int \frac{e^{\arctan y}}{1+y^2}\,dy$ u-sub or trig sub
8. $\int \frac{1}{x(x^2+4)}\,dx$ partial fractions or trig sub
9. $\int \cos\theta\sin^3\theta\,d\theta$ trig identities & u-sub
10. $\int \frac{\sqrt{4-x^2}}{x}\,dx$ trig sub
11. $\int 2x\arctan x\,dx$ IBP
12. $\int \frac{1}{(x-2)(x^2+4)}\,dx$ partial fractions

FULL SOLUTIONS FOLLOW

1. $\int xe^{2x}\,dx$

$u = x \quad dv = e^{2x}\,dx$

$du = dx \quad v = \frac{e^{2x}}{2}$

$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}$

$\boxed{= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$

2. $\int \tan^3\theta \sec\theta \, d\theta = \int \tan^2\theta \sec\theta \tan\theta \, d\theta$

$u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$= \int (\sec^2\theta - 1) \sec\theta \tan\theta \, d\theta$

$= \int (u^2 - 1)\, du$

$= \frac{u^3}{3} - u + C = \boxed{\frac{\sec^3\theta}{3} - \sec\theta + C}$

3. $\int x\sqrt{x^2+1}\,dx$

$u = x^2 + 1$

$du = 2x\,dx$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$\boxed{= \frac{1}{3}(x^2+1)^{3/2} + C}$

4. $\int \cos^2(3\theta)\,d\theta = \int \frac{1}{2}(1+\cos 6\theta)\,d\theta = \boxed{\frac{\theta}{2} + \frac{\sin 6\theta}{12} + C}$

5. $\int \frac{4}{4-x^2}\,dx = \int \frac{4}{(2-x)(2+x)}\,dx = \int \frac{A}{2-x} + \frac{B}{2+x}\,dx$

$= \int \frac{1}{2-x} + \frac{1}{2+x}\,dx$

$\boxed{= -\ln|2-x| + \ln|2+x| + C}$

$4 = A(2+x) + B(2-x)$

$4 = A2 + Ax + B2 - Bx$

$A - B = 0 \Rightarrow A = B$

$2A + 2B = 4 \Rightarrow B = 1,\ A = 1$

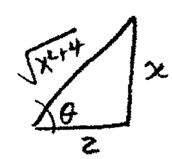
6. $\int \frac{1}{\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$= \int \frac{2\sec^2\theta}{\sqrt{4(\tan^2\theta+1)}}\,d\theta = \int \frac{\cancel{2}\sec^2\theta}{\cancel{2}\sec\theta}\,d\theta$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$



$\boxed{= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$

7. $\int \frac{e^{\arctan y}}{1+y^2}\,dy$ $\quad u = \arctan y$, $du = \frac{1}{1+y^2}\,dy$ $\quad = \int e^u\,du = \boxed{e^{\arctan y} + C}$

8. $\int \frac{1}{x(x^2+4)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\,dx$

$1 = Ax^2 + 4A + Bx^2 + Cx$

$A+B=0$
$C=0$
$4A=1 \Rightarrow A = 1/4,\ B = -1/4$

$= \int \frac{1/4}{x} + \frac{-1/4\,x}{x^2+4}\,dx$

$= 1/4 \ln|x| - 1/4 \int \frac{x}{x^2+4}\,dx$ $\quad u = x^2+4$, $du = 2x\,dx$

$\boxed{= 1/4 \ln|x| - \frac{1}{8}\ln|x^2+4| + C}$

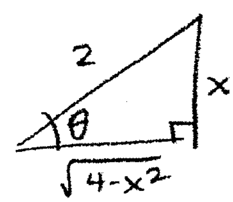
9. $\int \cos\theta \sin^3\theta\,d\theta$ $\quad u = \sin\theta$, $du = \cos\theta\,d\theta$ $\quad = \int u^3\,du = \frac{u^4}{4} + C = \boxed{\frac{\sin^4\theta}{4} + C}$

10. $\int \frac{\sqrt{4-x^2}}{x}\,dx$ $\quad x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$ $\quad = \int \frac{\sqrt{4-4\sin^2\theta}}{2\sin\theta} \cdot 2\cos\theta\,d\theta$

$= \int \frac{2\cos\theta}{2\sin\theta} \cdot 2\cos\theta\,d\theta = \int \frac{2\cos^2\theta}{\sin\theta}\,d\theta = \int 2\frac{(1-\sin^2\theta)}{\sin\theta}\,d\theta$

$= \int \frac{2}{\sin\theta} - 2\sin\theta\,d\theta = \int 2\csc\theta - 2\sin\theta\,d\theta$

$= \underbrace{2\ln|\csc\theta - \cot\theta|}_{\text{from back of book}} + 2\cos\theta + C$



$\boxed{= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C}$

11. $\int 2x\arctan x\,dx = x^2\arctan x - \underbrace{\int \frac{x^2}{1+x^2}\,dx}_{\text{add 0 in denom.}}$

$u = \arctan x$ $\quad dv = 2x\,dx$
$du = \frac{1}{1+x^2}\,dx$ $\quad v = x^2$

$= x^2\arctan x - \int \frac{x^2+1}{x^2+1} + \frac{-1}{x^2+1}\,dx = x^2\arctan x - x + \arctan x + C$

12. $\int \frac{1}{(x-2)(x^2+4)}\,dx = \int \frac{A}{(x-2)} + \frac{Bx+C}{x^2+4}\,dx$

$1 = Ax^2 + 4A + Bx^2 - 2Bx + Cx - 2C$

$1 = Ax^2 + Bx^2 + Cx - 2Bx + 4A - 2C$

$0 = A+B \Rightarrow A = -B$

$0 = C - 2B \Rightarrow C = 2B$

$1 = 4A - 2C \Rightarrow 1 = -4B - 4B$

$1 = -8B$

$B = -1/8$

so $A = 1/8$, $C = -1/4$

$\int \frac{1/8}{(x-2)} + \frac{-1/8x - 1/4}{x^2+4}\,dx = \frac{1}{8}\ln|x-2| + \int \frac{-1/8x}{x^2+4} + \frac{-1/4}{x^2+4}\,dx$

u-sub ($u = x^2+4$, $du = 2x\,dx$) arctanx form

$$\boxed{= \frac{1}{8}\ln|x-2| - \frac{1}{16}\ln|x^2+4| - \frac{1}{8}\arctan(x/2) + C}$$