Extra Credit Exam 3

1. From the exam, recall that $\dfrac{3x+2}{x^3+3x^2+2x} = \dfrac{1}{x} + \dfrac{1}{x+1} - \dfrac{2}{x+2}$

$$\sum_{n=2}^{\infty} \frac{3n+2}{n^3+3n^2+2n} = \sum_{n=2}^{\infty} \left(\frac{1}{n} + \frac{1}{n+1} - \frac{2}{n+2} \right)$$

a) $$S_5 = \left(\frac{1}{2} + \frac{1}{3} - \cancel{\frac{2}{4}}\right) + \left(\frac{1}{3} + \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}}\right) + \left(\cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} - \frac{2}{6}\right) + \left(\cancel{\frac{1}{5}} + \frac{1}{6} - \frac{2}{7}\right)$$

$$= \underbrace{\frac{1}{2} + \frac{1}{3} + \frac{1}{3}} \underbrace{- \frac{2}{6} + \frac{1}{6} - \frac{2}{7}}$$

$$S_6 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \cancel{\frac{2}{6}} + \cancel{\frac{1}{6}} - \frac{2}{7} + \left(\cancel{\frac{1}{6}} + \frac{1}{7} - \frac{2}{8}\right)$$

$$= \underbrace{\frac{1}{2} + \frac{1}{3} + \frac{1}{3}} \underbrace{- \frac{2}{7} + \frac{1}{7} - \frac{2}{8}}$$

$$S_N = \left(\frac{1}{2} + \frac{1}{3} - \frac{2}{4}\right) + \left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{2}{6}\right) + \left(\frac{1}{5} + \frac{1}{6} - \frac{2}{7}\right) + \left(\frac{1}{6} + \frac{1}{7} - \frac{2}{8}\right)$$

$$+ \dots + \underbrace{\left(\cancel{\frac{1}{N-3}} + \cancel{\frac{1}{N-2}} - \cancel{\frac{2}{N-1}}\right)}_{N-3} + \underbrace{\left(\cancel{\frac{1}{N-2}} + \cancel{\frac{1}{N-1}} - \cancel{\frac{2}{N}}\right)}_{N-2} + \underbrace{\left(\cancel{\frac{1}{N-1}} + \cancel{\frac{1}{N}} - \frac{2}{N+1}\right)}_{N-1 \text{ term}} + \underbrace{\left(\cancel{\frac{1}{N}} + \frac{1}{N+1} - \frac{2}{N+2}\right)}_{N^{th} \text{ term}}.$$

$$S_N = \underbrace{\frac{1}{2} + \frac{1}{3} + \frac{1}{3}} \underbrace{- \frac{2}{N+1} + \frac{1}{N+1} - \frac{2}{N+2}}$$

b) $$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{2}{N+1} + \frac{1}{N+1} - \frac{2}{N+2} \right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{3}$$

$$= \frac{7}{6}$$

2. $\sum_{n=1}^{\infty} \cos(\pi n)\sin(1/n) = \sum_{n=1}^{\infty} (-1)^n \sin(1/n)$

a) Look at $\sum_{n=1}^{\infty} \left|(-1)^n \sin(1/n)\right| = \sum_{n=1}^{\infty} \sin(1/n)$

$a_n = \sin(1/n) > 0$, $b_n = 1/n > 0$

<u>LCT</u>:

$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \sin(1/n)$ diverges by Limit Comp. Test.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$ is <u>not</u> <u>absolutely</u> <u>conv.</u>

b) $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$

<u>AST</u>

1) $\sin(1/n) > 0$

2) $\lim_{n\to\infty} \sin(1/n) = \sin 0 = 0$

3) $f(x) = \sin(1/x)$

$f'(x) = -\frac{\cos(1/x)}{x^2} < 0$

so $f(x)$ is decreasing.

Thus, $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$ converges by the Alternating Series Test.

By a) and b), $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$ is conditionally convergent.

3. a) $\displaystyle\int_N^{N+1} (t-N)e^{-st}\,dt$

let $x = t-N$
$dx = dt$
when $t = N$, $x = 0$
$t = N+1$, $x = 1$

$$= \int_0^1 x e^{-s(x+N)}\,dx$$

$$= \int_0^1 x e^{-sx} \underbrace{e^{-sN}}_{\text{constant}}\,dx = e^{-sN}\int_0^1 x e^{-sx}\,dx$$

IBP
$u = x \quad dv = e^{-sx}dx$
$du = dx \quad v = -\dfrac{e^{-sx}}{s}$

$$= e^{-sN}\left[-\frac{xe^{-sx}}{s}\Big|_0^1 + \int_0^1 \frac{e^{-sx}}{s}\,dx\right]$$

$$= e^{-sN}\left[-\frac{e^{-s}}{s} + \left(-\frac{e^{-sx}}{s^2}\Big|_0^1\right)\right] = e^{-sN}\underbrace{\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]}_{\text{doesn't depend on } N}$$

b) $\displaystyle F_N(s) = \int_0^{N+1} f_N(t)e^{-st}\,dt$ where $f_N(t)$ is defined on the sheet.

$$= \int_0^1 t e^{-st}\,dt + \int_1^2 (t-1)e^{-st}\,dt + \int_2^3 (t-2)e^{-st}\,dt$$

$$+ \dots + \int_N^{N+1} (t-N)e^{-st}\,dt.$$

$$= e^0\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right] + e^{-s}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right] + e^{-2s}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$+ \dots + e^{-sN}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right].$$

$$= \left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left(1 + e^{-s} + e^{-2s} + \cdots + e^{-Ns}\right)$$

$$= \left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left(\left(\frac{1}{e^s}\right)^0 + \left(\frac{1}{e^s}\right)^1 + \left(\frac{1}{e^s}\right)^2 + \cdots + \left(\frac{1}{e^s}\right)^N\right)$$

$$= \sum_{n=0}^{N} \underbrace{\left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]}_{c} \underbrace{\left(\frac{1}{e^s}\right)}_{r}^n$$

c) $$F(s) = \lim_{N\to\infty} \sum_{n=0}^{N} \left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left(\frac{1}{e^s}\right)^n$$

$$= \frac{\left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]}{1 - \frac{1}{e^s}}$$

because geometric series with $|r| = 1/e^s < 1$ (since $s > 0$)

d) $$\frac{\left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]}{1 - \frac{1}{e^s}} = \left[\frac{-1}{se^s} - \frac{1}{s^2e^s} + \frac{1}{s^2}\right]\left(\frac{e^s}{e^s - 1}\right)$$

$$= \left[\frac{-s - 1 + e^s}{s^2 e^s}\right]\left(\frac{e^s}{1 - e^s}\right) = \frac{e^s - 1 - s}{s^2(e^s - 1)} \quad \checkmark$$