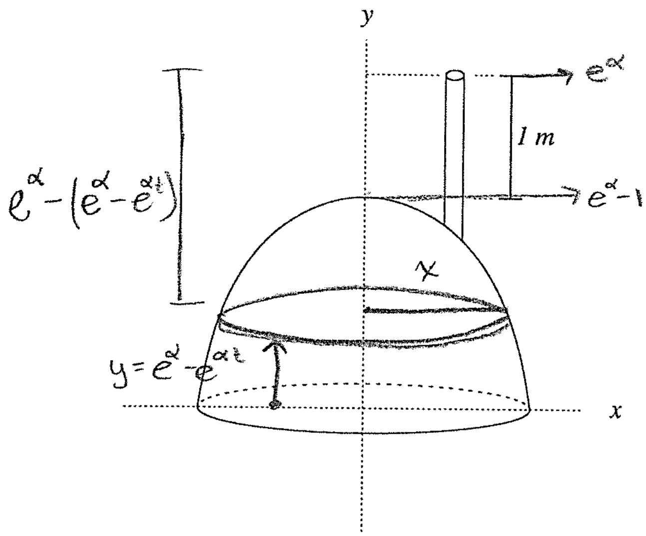
Final Exam Extra Credit



For $0 \le t \le 1$,

- $x(t) = \sqrt{2\sin t\cos t + t^2}$
- $y(t) = e^{\alpha} - e^{\alpha t}$ $\quad (\alpha > 0)$

$y'(t) = -\alpha e^{\alpha t}$

a)

$$A_{\text{slice}} = \pi x^2 = \pi(2\sin t\cos t + t^2)$$

$$V_{\text{slice}} = \pi(2\sin t\cos t + t^2)\,\Delta y \qquad \Delta y = y'(t)\,\Delta t$$

$$= \pi(2\sin t\cos t + t^2)(-\alpha e^{\alpha t})\,\Delta t$$

$$W_{\text{slice}} = \pi\rho g(2\sin t\cos t + t^2)(-\alpha e^{\alpha t})(e^{\alpha t})\,\Delta t$$

$$W = \int_0^1 \pi\rho g(2\sin t\cos t + t^2)(-\alpha e^{2\alpha t})\,dt$$

b) $W = \int_0^1 \pi\rho g(2\sin t\cos t + t^2)(-\alpha e^{2\alpha t})\, dt$

Note that $2\sin t\cos t = \sin 2t$ by identity.

$$= \int_0^1 -\alpha\pi\rho g\,(\sin 2t + t^2)\, e^{2\alpha t}\, dt$$

$$= -\alpha\pi\rho g \int_0^1 e^{2\alpha t}\sin 2t + t^2 e^{2\alpha t}\, dt$$

$$= -\alpha\pi\rho g\left[\underbrace{\int_0^1 e^{2\alpha t}\sin 2t\, dt}_{\text{integral A}} + \underbrace{\int_0^1 t^2 e^{2\alpha t}\, dt}_{\text{integral B}}\right]$$

Integral A:

$$\int_0^1 e^{2\alpha t}\sin 2t\, dt$$

IBP

$u = \sin 2t \quad dv = e^{\alpha t}dt$

$du = 2\cos 2t\, dt \quad v = \frac{e^{2\alpha t}}{2\alpha}$

$$= \frac{e^{2\alpha t}\sin 2t}{2\alpha}\Big|_0^1 - \frac{1}{\alpha}\int_0^1 e^{2\alpha t}\cos 2t\, dt$$

IBP again

$u = \cos 2t \quad dv = e^{2\alpha t}dt$

$du = -2\sin 2t\, dt \quad v = \frac{e^{2\alpha t}}{2\alpha}$

So we have:

$$\int_0^1 e^{2\alpha t}\sin 2t\, dt = \frac{e^{2\alpha}\sin 2}{2\alpha} - \frac{1}{\alpha}\left[\frac{e^{2\alpha t}\cos 2t}{2\alpha}\Big|_0^1 + \int_0^1 + \frac{2e^{2\alpha t}}{2\alpha}\sin 2t\, dt\right]$$

$$\Rightarrow \int_0^1 e^{2\alpha t}\sin 2t\, dt = \frac{e^{2\alpha}\sin 2}{2\alpha} - \frac{e^{2\alpha}\cos 2}{2\alpha^2} + \frac{1}{2\alpha^2} - \frac{1}{\alpha^2}\int_0^1 e^{2\alpha t}\sin 2t\, dt$$

$\boxed{+\frac{1}{\alpha^2}\int_0^1 e^{2\alpha t}\sin 2t\, dt}$ add over $+\frac{1}{\alpha^2}\int_0^1 e^{2\alpha t}\sin 2t\, dt$

$$\Rightarrow \quad \frac{1+\alpha^2}{\alpha^2}\int_0^1 e^{2\alpha t}\sin 2t\,dt = \frac{e^{2\alpha}\sin 2}{2\alpha} - \frac{e^{2\alpha}\cos 2}{2\alpha^2} + \frac{1}{2\alpha^2}$$

$$\text{So, } \int_0^1 e^{\alpha t}\sin 2t\,dt = \frac{\alpha^2}{1+\alpha^2}\left[\frac{e^{2\alpha}\sin 2}{2\alpha} - \frac{e^{2\alpha}\cos 2}{2\alpha^2} + \frac{1}{2\alpha^2}\right]$$

$$= \frac{1}{1+\alpha^2}\left[\frac{\alpha e^{2\alpha}\sin 2}{2} - \frac{e^{2\alpha}\cos 2}{2} + \frac{1}{2}\right] \text{ J}$$

Integral B:

$$\int_0^1 t^2 e^{2\alpha t}\,dt$$

IBP

$u = t^2 \qquad dv = e^{2\alpha t}\,dt$

$du = 2t\,dt \qquad v = \frac{e^{2\alpha t}}{2\alpha}$

$$= \frac{t^2 e^{2\alpha t}}{2\alpha}\Big|_0^1 - \int_0^1 \frac{1}{\alpha} t e^{2\alpha t}\,dt$$

IBP again

$u = t \qquad dv = e^{2\alpha t}\,dt$

$du = dt \qquad v = \frac{e^{2\alpha t}}{2\alpha}$

$$= \left(\frac{e^{2\alpha}}{2\alpha} - 0\right) - \frac{1}{\alpha}\left[\frac{t e^{2\alpha t}}{2\alpha}\Big|_0^1 - \int_0^1 \frac{e^{2\alpha t}}{2\alpha}\,dt\right]$$

$$= \frac{e^{2\alpha}}{2\alpha} - \frac{1}{\alpha}\left[\left(\frac{e^{2\alpha}}{2\alpha} - 0\right) - \left(\frac{e^{2\alpha t}}{4\alpha^2}\Big|_0^1\right)\right]$$

$$= \frac{e^{2\alpha}}{2\alpha} - \frac{e^{2\alpha}}{2\alpha^2} + \frac{1}{\alpha}\left[\frac{e^{2\alpha}}{4\alpha^2} - \frac{1}{4\alpha^2}\right] = \frac{e^{2\alpha}}{2\alpha} - \frac{e^{2\alpha}}{2\alpha^2} + \frac{e^{2\alpha}}{4\alpha^3} - \frac{1}{4\alpha^3}$$

$$= \frac{1}{4\alpha^3}\left[2\alpha^2 e^{2\alpha} - 2\alpha e^{2\alpha} + e^{2\alpha} - 1\right] \text{ J}$$

Total work = integral A + integral B

$$= -\pi\rho g\alpha\left(\frac{1}{2(1+\alpha^2)}\left[\alpha e^{2\alpha}\sin 2 - e^{2\alpha}\cos 2 + 1\right] + \frac{1}{4\alpha^3}\left[2\alpha^2 e^{2\alpha} - 2\alpha e^{2\alpha} + e^{2\alpha} - 1\right]\right) \text{ J}$$