10.7

① $f(x) = \cos x, \quad c = \pi/3$

a)

n	$f^{(n)}(x)$	$f^{(n)}(\pi/3)$
0	$\cos x$	$1/2$
1	$-\sin x$	$-\sqrt{3}/2$
2	$-\cos x$	$-1/2$
3	$\sin x$	$\sqrt{3}/2$
4	$\cos x$	$1/2$
$\vdots$		$\vdots$

$$f(x) = \frac{1}{2} + \frac{-\sqrt{3}}{2}(x-\pi/3) + \frac{-1}{2\cdot 2!}(x-\pi/3)^2$$

$$+ \frac{\sqrt{3}}{2\,3!}(x-\pi/3)^3 + \frac{1}{2\cdot 4!}(x-\pi/3)^4 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/3)^{2n}}{2(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\sqrt{3}\,(x-\pi/3)^{2n+1}}{2(2n+1)!}$$

b) $f(x) = \cos x = \cos((x-\pi/3)+\pi/3)$

$$= \cos(x-\pi/3)\cos\pi/3 - \sin(x-\pi/3)\sin\pi/3$$

$$= \cos(x-\pi/3)\cdot\frac{1}{2} - \sin(x-\pi/3)\frac{\sqrt{3}}{2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/3)^{2n}}{2(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}\,(x-\pi/3)^{2n+1}}{2(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/3)^{2n}}{2(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\sqrt{3}\,(x-\pi/3)^{2n+1}}{2(2n+1)!}$$ ✓

(2) $r(x) = \dfrac{3}{(1-x)(1+2x)}$

Aside:

$$\frac{3}{(1-x)(1+2x)} = \frac{A}{1-x} + \frac{B}{1+2x}$$

$$3 = A(1+2x) + B(1-x)$$

let $x=1$: $3 = A(3)$ $\quad A =$

let $x = -1/2$: $3 = B(3/2)$ $\quad B = 2$

$$r(x) = \frac{1}{1-x} + \frac{2}{1+2x}$$

$$r(x) = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2(-2x)^n$$

if $|x|<1$ $\qquad$ if $|x|<1/2$

$$r(x) = \sum_{n=0}^{\infty} (1 + 2^{n+1}(-1)^n)\, x^n$$ provided $|x| < 1/2$.

(3) $f(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$

a) $$f(x) = \int_0^x \frac{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}}{t}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \Bigg|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} - 0.$$

b) $$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

Because $f(1)$ is a convergent alternating series:

$$|f(1) - S_N| < b_{N+1} \leq 0.0001$$

$$S_N = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \frac{1}{9\cdot 9!} \cdots \pm \frac{1}{(2N+1)(2N+1)!}$$

$$S_N = 1 - \frac{1}{18} + \frac{1}{600} - \underbrace{\frac{1}{35280}}$$

this term won't affect the sum in the first 3 decimal places

So, $|s - S_2| < b_3 \leq 0.0001$, which

means that $S_2 = 1 - \frac{1}{18} + \frac{1}{600} = 0.946111$

is within 3 decimal places of $f(1)$.

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b) $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$

c) $e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots\right)$

d) $e^{i\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta}$

e) $e^{i\theta} = \cos\theta + i\sin\theta$

f) $e^{i\pi} = \cos\pi + i\sin\pi$

$e^{i\pi} = -1 + 0$

$\boxed{e^{i\pi} + 1 = 0}$ ✓