7.3 problems

complete the square

$$x^2-2x-3 = \underbrace{x^2-2x+(-1)^2}_{(x-1)^2}-(-1)^2-3$$

$$= (x-1)^2-4$$

1. $$\int \frac{dx}{(x^2-2x-3)^{3/2}}$$

$$= \int \frac{dx}{((x-1)^2-4)^{3/2}} \qquad \begin{aligned} u &= x-1 \\ du &= dx \end{aligned}$$

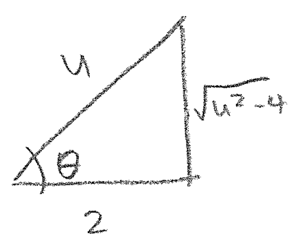
$$= \int \frac{du}{(u^2-4)^{3/2}} \qquad \begin{aligned} u &= 2\sec\theta \\ du &= 2\sec\theta\tan\theta\,d\theta \end{aligned}$$

$$= \int \frac{2\sec\theta\tan\theta}{(4\sec^2\theta-4)^{3/2}}\,d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{(4\tan^2\theta)^{3/2}}\,d\theta = \int \frac{2\sec\theta\,\cancel{\tan\theta}}{8\tan^{\cancel{3}2}\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{w^2}\,dw \qquad \begin{aligned} w &= \sin\theta \\ dw &= \cos\theta\,d\theta \end{aligned}$$

$$= -\frac{1}{4}\cdot\frac{1}{w}+C = -\frac{1}{4\sin\theta}$$



$$\frac{u}{2} = \sec\theta$$

$$= -\frac{u}{4\sqrt{u^2-4}}+C$$

$$= \frac{1-x}{4\sqrt{(x-1)^2-4}}+C$$

2. $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$

$x = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

when $x = \sqrt{3}$, $1 = \sec\theta \Rightarrow \theta = 0$

$x = 2$, $2 = \sqrt{3}\sec\theta \Rightarrow \frac{2}{\sqrt{3}} = \frac{1}{\cos\theta}$

$\Rightarrow \theta = \pi/6$

$$= \int_0^{\pi/6} \frac{\sqrt{3\sec^2\theta-3}}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/6} \sqrt{3}\tan\theta\cdot\tan\theta\,d\theta$$

$$= \int_0^{\pi/6} \sqrt{3}\,(\sec^2\theta-1)\,d\theta = \sqrt{3}\,(\tan\theta-\theta)\Big|_0^{\pi/6}$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}}-\pi/6\right) = 1-\frac{\sqrt{3}\,\pi}{6}$$

3. $\displaystyle\int \frac{dx}{x\sqrt{x^2+x}}$

a) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2}} = \int \frac{dx}{x\sqrt{(x+\tfrac{1}{2})^2-\tfrac{1}{4}}}$

let $x+\tfrac{1}{2} = \tfrac{1}{2}\sec\theta$

$dx = \tfrac{1}{2}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\tfrac{1}{2}\sec\theta\tan\theta\, d\theta}{(\tfrac{1}{2}\sec\theta-\tfrac{1}{2})\sqrt{\tfrac{1}{4}\sec^2\theta-\tfrac{1}{4}}}$$

$$= \int \frac{\tfrac{1}{2}\sec\theta\tan\theta}{(\tfrac{1}{2}\sec\theta-\tfrac{1}{2})\,\tfrac{1}{2}\tan\theta}\, d\theta = \int \frac{\sec\theta}{(\tfrac{1}{2}\sec\theta-\tfrac{1}{2})}\cdot\frac{(\tfrac{1}{2}\sec\theta+\tfrac{1}{2})}{(\tfrac{1}{2}\sec\theta+\tfrac{1}{2})}\, d\theta$$

$$= \int \frac{\tfrac{1}{2}\sec^2\theta+\tfrac{1}{2}\sec\theta}{\tfrac{1}{4}\sec^2\theta-\tfrac{1}{4}}\, d\theta = \frac{1}{2}\cdot 4\int \frac{\sec^2\theta+\sec\theta\, d\theta}{\tan^2\theta}$$

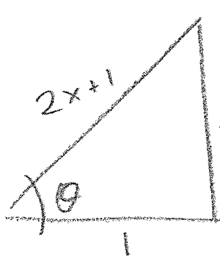
$$= 2\int \frac{1}{\cos^2\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta + 2\int \frac{1}{\cos\theta}\,\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= 2\int \csc^2\theta\, d\theta + 2\int \frac{1}{u^2}\, du$$

$$= -2\cot\theta - \frac{2}{u} + C = -2(\cot\theta + \csc\theta) + C$$



$\sqrt{(2x+1)^2-1}$

$= \sqrt{4x^2+4x}$

$= 2\sqrt{x^2+x}$

$x+\tfrac{1}{2} = \tfrac{1}{2}\sec\theta$

$2x+1 = \sec\theta$

$$= -2\left(\frac{1}{2\sqrt{x^2+x}} + \frac{2x+1}{2\sqrt{x^2+x}}\right) + C$$

$$= -\frac{(2x+2)}{\sqrt{x^2+x}} + C = \frac{-2(x+1)}{\sqrt{x^2+x}} + C$$

b) $\displaystyle\int \frac{\frac{1}{x^2}\,dx}{\sqrt{1+1/x}}$ $\quad u = 1 + 1/x$

$du = -1/x^2\,dx$

$$= \int \frac{-\,du}{\sqrt{u}} = -2\sqrt{u} + C = -2\sqrt{1+1/x} + C$$

c) Show $-2\sqrt{1+1/x} = \dfrac{-2(x+1)}{\sqrt{x^2+x}}$

$$-2\sqrt{1+1/x} = -2\sqrt{1+1/x}\left(\frac{\sqrt{1+1/x}}{\sqrt{1+1/x}}\right) = \frac{-2(1+1/x)}{\sqrt{1+1/x}}\cdot\left(\frac{x}{x}\right)$$

$$= \frac{-2(x+1)}{x\sqrt{1+1/x}} = \frac{-2(x+1)}{\sqrt{x^2+x}} \quad \checkmark$$