7.5 problems

1. $$\int \frac{2x^3+9x^2+19x+20}{x^3+3x^2+7x+5}\,dx$$

long division:
$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad 2 \\ x^3+3x^2+7x+5 & 2x^3+9x^2+19x+20 \\ & -(2x^3+6x^2+14x+10) \\ \hline & \quad\quad 3x^2+5x+10 \end{array}$$

partial fractions: $$\frac{3x^2+5x+10}{(x+1)(x^2+2x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+5}$$

$$3x^2+5x+10 = A(x^2+2x+5) + (Bx+C)(x+1)$$

- let $x=-1$: $3-5+10 = A(1-2+5)$

  $8 = A4$

  $\Rightarrow A=2$

- $3x^2+5x+10 = 2(x^2+2x+5) + (Bx+C)(x+1)$

  $3x^2+5x+10 = 2x^2+4x+10 + Bx^2 + Bx + Cx + C$

  $3x^2+5x+10 = (2+B)x^2 + (4+B+C)x + C+10$

$$\left[\begin{array}{ll} 2+B=3 & \Rightarrow B=1 \\ 4+B+C=5 & \Rightarrow C=0 \\ C+10=10 & \checkmark \end{array}\right.$$

$$\int \frac{2x^3+9x^2+19x+20}{x^3+3x^2+7x+5}\,dx = \int 2 + \frac{2}{x+1} + \frac{x+0}{x^2+2x+5}\,dx$$

1 ctd)

$$\int 2 + \frac{2}{x+1} + \frac{x}{x^2+2x+5}\,dx = 2\ln|x+1| + \int \frac{x}{(x+1)^2+4}\,dx$$

(via complete the square)

$u = x+1$
$du = dx$

$$= 2x + 2\ln|x+1| + \int \frac{u-1}{u^2+4}\,du$$

$$= 2x + 2\ln|x+1| + \int \frac{u}{u^2+4}\,du - \int \frac{1}{u^2+4}\,du$$

$w = u^2+4$
$dw = 2u\,du$

$$= 2x + 2\ln|x+1| + \frac{1}{2}\ln|(x+1)^2+4| - \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

2. a) $\int \frac{dt}{1+e^{2t}}$

$u = 1+e^{2t} \iff u-1 = e^{2t}$
$du = 2e^{2t}\,dt$
$\frac{du}{2(u-1)} = dt$

$$= \frac{1}{2}\int \frac{du}{u(u-1)} = \frac{1}{2}\int \frac{A}{u} + \frac{B}{u-1}\,du$$

$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + Bu$

- let $u=1$: $1 = B$
- let $u=0$: $-1 = A$

$$= \frac{1}{2}\int \frac{-1}{u} + \frac{1}{u-1}\,du$$

$$= -\frac{1}{2}\ln|1+e^{2t}| + \frac{1}{2}\ln|e^{2t}| + C$$

$$= \frac{1}{2}\ln\left(\frac{e^{2t}}{1+e^{2t}}\right) + C$$

2b) $\int \frac{1}{1+e^{2t}} \cdot \frac{e^{-2t}}{e^{-2t}}\,dt = \int \frac{e^{-2t}}{e^{-2t}+1}\,dt$

$u = e^{-2t}+1$

$du = -2e^{-2t}\,dt$

$= -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|e^{-2t}+1| + C$

c) From a): $\frac{1}{2}\ln\left(\frac{e^{2t}}{e^{2t}+1}\right) = -\frac{1}{2}\ln\left(\frac{e^{2t}+1}{e^{2t}}\right)$

$= -\frac{1}{2}\ln\left(1+\frac{1}{e^{2t}}\right) = -\frac{1}{2}\ln(1+e^{-2t})$ ✓

3. $\int \frac{\cos x}{1-\sin^2 x}\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$ $= \int \frac{1}{1-u^2}\,du = \int \frac{A}{1-u} + \frac{B}{1+u}\,du$

$= \int \frac{1/2}{1-u} + \frac{1/2}{1+u}\,du$

$w = 1-u$

$dw = -du$

$= -\frac{1}{2}\ln|1-\sin x| + \frac{1}{2}\ln|1+\sin x| + C$

$\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$

$1 = A(1+u) + B(1-u)$

let $u=1$: $A = 1/2$

let $u=-1$: $B = 1/2$

Equivalent:

$\ln|\sec x + \tan x| = \ln\left|\frac{1+\sin x}{\cos x}\right| = \ln|1+\sin x| - \ln|\cos x|$

$= \ln|1+\sin x| - \frac{1}{2}\ln|1-\sin^2 x| = \ln|1+\sin x| - \frac{1}{2}\ln|(1-\sin x)(1+\sin x)|$

$= \ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x| - \frac{1}{2}\ln|1+\sin x|$

$= \frac{1}{2}\ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x|$ ✓