Math 182 | Problems | Name: Key
Due: 27 Feb 2018 | Section: 7.7 | Point Values in $\boxed{\text{boxes}}$.

1. $\boxed{1}$ Find the partial fraction decomposition of

$$\frac{1}{x^2-x-6}.$$

$$\frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$= \frac{1/5}{x-3} + \frac{-1/5}{x+2}$$

$1 = A(x+2) + B(x-3)$

let $x=3$: $1 = A(5)$, $A = 1/5$ let $x=-2$: $1 = B(-5)$, $B = -1/5$

2. Evaluate the following, or show that they diverge.

(a) $\boxed{1.5}$ $$\int_0^4 \frac{dx}{x^2-x-6} = \int_0^3 \frac{1/5}{x-3} + \frac{-1/5}{x+2}\,dx + \int_3^4 \frac{1/5}{x-3} + \frac{-1/5}{x+2}\,dx$$

Start with one: $$\int_0^3 \frac{1/5}{x-3} + \frac{-1/5}{x+2}\,dx = \lim_{R\to 3^-}\int_0^R \frac{1/5}{x-3} + \frac{-1/5}{x+2}\,dx = \lim_{R\to 3^-} \frac{1}{5}\ln\left|\frac{x-3}{x+2}\right|\Bigg|_0^R$$

$$= \lim_{R\to 3^-} \frac{1}{5}\ln\left|\frac{R-3}{R+2}\right| - \frac{1}{5}\ln\left|\frac{-3}{2}\right| = -\infty \quad \therefore \boxed{\int_0^4 \frac{dx}{x^2-x-6} \text{ diverges}}$$

(b) $\boxed{1.5}$ $$\int_4^\infty \frac{dx}{x^2-x-6} = \lim_{R\to\infty} \frac{1}{5}\ln\left|\frac{x-3}{x+2}\right|\Bigg|_4^R \quad \text{(from a)}$$

$$= \lim_{R\to\infty} \frac{1}{5}\ln\left|\frac{R-3}{R+2}\right| - \frac{1}{5}\ln\left|\frac{1}{6}\right| \overset{L'H}{=} \lim_{R\to\infty} \underbrace{\frac{1}{5}\ln 1}_{0} - \frac{1}{5}\ln(1/6)$$

$$\boxed{= \frac{1}{5}\ln 6}$$

(c) $\boxed{2}$ $$\int_0^1 x\ln x\,dx = \lim_{R\to 0^+}\left[\frac{x^2\ln x}{2}\Bigg|_R^1 - \int_R^1 \frac{x}{2}\,dx\right]$$

IBP

$u = \ln x$ $dv = x\,dx$

$du = \frac{1}{x}dx$ $v = \frac{x^2}{2}$

$$= \lim_{R\to 0^+}\left[0 - \frac{R^2\ln R}{2} - \left(\frac{x^2}{4}\Bigg|_R^1\right)\right]$$

$$= \lim_{R\to 0^+}\left[\underbrace{-\frac{R^2\ln R}{2}}_{0} - \frac{1}{4} + \underbrace{\frac{R^2}{4}}_{0}\right]$$

$$= \boxed{-1/4}$$

Aside:

$$\lim_{R\to 0^+} \frac{-\ln R}{2R^{-2}} \overset{L'H}{=} \lim_{R\to 0^+} \frac{-\frac{1}{R}}{-4\frac{1}{R^3}}$$

$$= \lim_{R\to 0^+} \frac{R^2}{4} = 0$$

3. Use the Comparison Test to show the following converge or diverge.

(a) $\boxed{2}$ $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(3+x^3)} = \underbrace{\int_0^1 \frac{dx}{3\sqrt{x}+x^{7/2}}}_{①} + \underbrace{\int_1^\infty \frac{dx}{3\sqrt{x}+x^{7/2}}}_{②}$

① $0 \le \frac{1}{3\sqrt{x}+x^{7/2}} \le \frac{1}{3\sqrt{x}}$

Since $\frac{1}{3}\int_0^1 \frac{1}{x^{1/2}}\,dx$ converges ($p = 1/2 < 1$), then by Comp. Test, $\int_0^1 \frac{dx}{3\sqrt{x}+x^{7/2}}$ converges.

② $0 \le \frac{1}{3\sqrt{x}+x^{7/2}} \le \frac{1}{x^{7/2}}$

Since $\int_1^\infty \frac{1}{x^{7/2}}\,dx$ converges ($p = 7/2 > 1$), then by Comp. Test, $\int_1^\infty \frac{dx}{3\sqrt{x}+x^{7/2}}$ converges

Because each integral converges, $\int_0^\infty \frac{dx}{3\sqrt{x}+x^{7/2}}$ converges.

(b) $\boxed{2}$ $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx$ by symmetry (even function)

$$= 2\Big[\underbrace{\int_0^1 e^{-x^2}\,dx}_{①} + \underbrace{\int_1^\infty e^{-x^2}\,dx}_{②}\Big]$$

① on $[0,1]$, $0 \le e^{-x^2} \le 1$

so $\int_0^1 e^{-x^2}\,dx \le \int_0^1 1\,dx = 1$, a finite value

② on $[1,\infty)$, $x^2 \ge x$ for all x

so $-x^2 \le -x$

$\Rightarrow 0 \le e^{-x^2} \le e^{-x}$

$\int_1^\infty e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_1^R = \lim_{R\to\infty}(-e^{-R}+e^{-1}) = \frac{1}{e}$

Since $\int_1^\infty e^{-x}\,dx$ converges, by comp Test, $\int_1^\infty e^{-x^2}\,dx$ converges.

Because each integral converges, $\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx$ must converge.