**Math 182 Exam 1**
30 Jan 2018

**Name:** key
Point values in $\boxed{\text{boxes}}$.

1. Integrate.

(a) $\boxed{10}$ $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin\theta}\,d\theta$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

when $\theta = \pi/6$, $u = 1/2$
$\theta = \pi/2$, $u = 1$

$$= \int_{1/2}^{1} \frac{1}{u}\,du$$

$$= \ln u \Big|_{1/2}^{1}$$

$$= \ln 1 - \ln \tfrac{1}{2} = -\ln \tfrac{1}{2} = \ln 2$$

(b) $\boxed{15}$ $\displaystyle\int \frac{3+x}{\sqrt{1-9x^2}}\,dx = \int \frac{3}{\sqrt{1-9x^2}}\,dx + \int \frac{x}{\sqrt{1-9x^2}}\,dx$

$u = 3x$
$du = 3dx$

$w = 1-9x^2$
$dw = -18x\,dx$

$$= \int \frac{du}{\sqrt{1-u^2}} - \frac{1}{18}\int \frac{dw}{\sqrt{w}}$$

$$= \arcsin u - \frac{1}{18}\,2\sqrt{w} + C$$

$$= \arcsin(3x) - \frac{1}{9}\sqrt{1-9x^2} + C$$

2. $\boxed{10}$ In section 5.8 we saw that

$$\int \frac{dx}{x^2+1} = \arctan x + c. \tag{1}$$

Use an appropriate substitution and equation (1) to show

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c.$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a^2}\int \frac{du}{\frac{u^2}{a^2}+1} = \frac{1}{a^2}\int \frac{dy}{(u/a)^2+1}$$

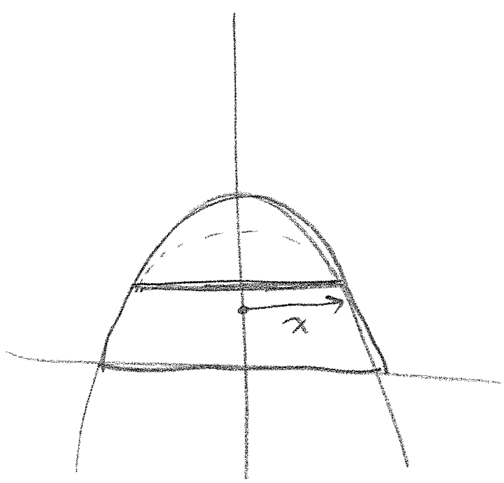
$w = u/a$

$dw = \frac{1}{a}\,du$

$$= \frac{a}{a^2}\int \frac{dw}{w^2+1} = \frac{1}{a}\arctan w + C$$

$$= \frac{1}{a}\arctan(u/a) + C$$

3. $\boxed{15}$ A solid has base bounded between the graph of $y = 4 - x^2$ and the $x$-axis. Cross sections perpendicular to the $y$-axis are semicircles. Find the volume of the solid.




① variable : $y$

② $A_{slice} = \frac{\pi r^2}{2} = \frac{\pi x^2}{2} = \frac{\pi}{2}(4-y)$

③ $V = \int_0^4 \frac{\pi}{2}(4-y)\,dy$

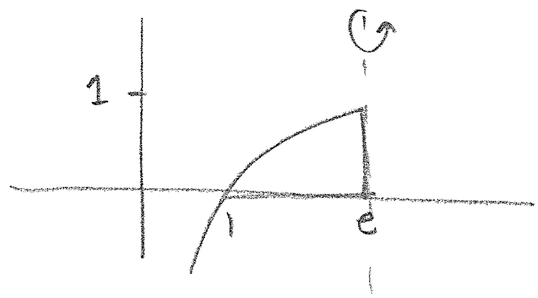
$$= \frac{\pi}{2}\left[4y - \frac{y^2}{2}\Big|_0^4\right]$$

$$= \frac{\pi}{2}\left[16-8\right] = 4\pi$$

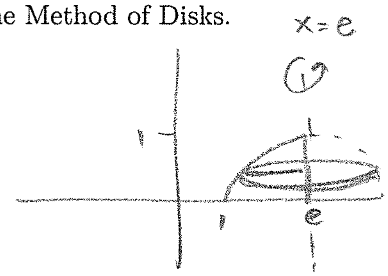
4. Consider the region bounded by the graph of $y = \ln x$, the $x$-axis and $x = e$. A solid is generated by rotating the region about the line $x = e$.

(a) $\boxed{4}$ Carefully sketch the region.

$y = \ln x \Leftrightarrow x = e^y$



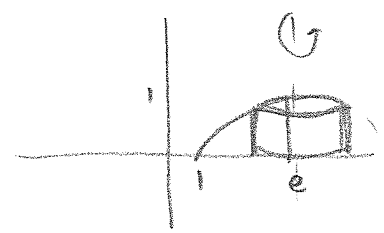

(b) $\boxed{8}$ Express the volume of the solid as an integral using the Method of Disks.




$$V = \int_0^1 \pi (e - e^y)^2 \, dy$$

(c) $\boxed{8}$ Express the volume of the solid as an integral using the Method of Shells.

$$V = \int_1^e 2\pi (e - x) \ln x \, dx$$




(d) $\boxed{10}$ Evaluate one of the integrals above to compute the volume.

disks (shells requires more advanced methods of integration than we have learned so far)

$$V = \int_0^1 \pi (e^2 - 2ee^y + e^{2y}) \, dy$$

$$= \pi \left[ e^2 y - 2ee^y + \frac{e^{2y}}{2} \Big|_0^1 \right]$$

$$= \pi \left[ e^2 - 2e^2 + \frac{e^2}{2} - \left(0 - 2e + \frac{1}{2}\right) \right]$$

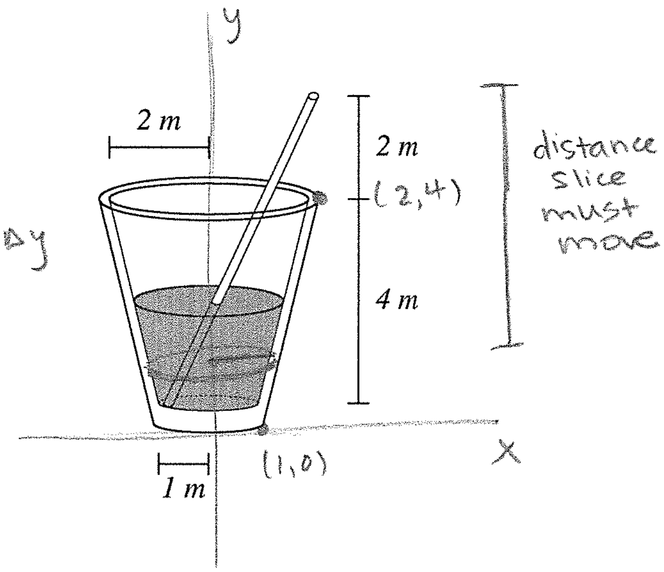
$$= \pi \left[ 2e - \frac{1}{2} - \frac{e^2}{2} \right]$$

5. $\boxed{20}$ After a long climb to the top of a beanstalk of height 800 m Jack sees a giant drinking a murky red cocktail. The interior of the glass is in the shape of a frustum with lower radius 1 m and upper radius 2 m. The height of the interior of the glass is 4 m. The Blood of an Englishman[1] cocktail she is drinking is very poorly mixed and has a density given by $\rho(y) = \kappa(12 - y)$ where $y$ measures the distance from the bottom of the interior of the glass. The giant is sipping her cocktail using a straw that extends 2 m past the top of the glass. When Jack arrived the giant had already consumed the 'top half' (as measured by height, not volume). How much work does the giant do by slurping the 'bottom half' up through the straw?

Express your solution as a definite integral; **do not evaluate the integral.** Use $g_g$ as the gravitational constant in the giant's fortress. Please start by clearly identifying the coordinate system that you will be using.

$$A_{slice} = \pi r^2 = \pi x^2 = \pi\left(\frac{y}{4}+1\right)^2$$

$$W_{slice} = \pi g\left(\kappa(12-y)\right)\left(\frac{y}{4}+1\right)^2(6-y)\,\Delta y$$




$$\boxed{W = \int_0^2 \pi g \kappa\,(12-y)\left(\frac{y}{4}+1\right)^2(6-y)\,dy}$$

Aside

$$m = \frac{4-0}{2-1} = \frac{4}{1} = 4$$

$$y = 4(x-1)$$

$$y = 4x-4$$

$$y+4 = 4x$$

$$x = \frac{y}{4}+1$$

| Problem | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Value | 25 | 10 | 15 | 30 | 20 | 100 |
| Points | | | | | | |

[1] No Englishmen were harmed in the writing of this question. No blood of any kind was used for her cocktail; the giant is a very pleasant vegan.