**Math 182 Exam 3** **Name:** Key

23 March 2018 Point values in $\boxed{\text{boxes}}$.

1. $\boxed{12}$ For each of the following, find the sum or show it diverges.

(a)

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

$$S_N = 1 - \frac{1}{N+1}$$

$$\lim_{N\to\infty} S_N = 1$$

(b)

$$\left(\arctan 1-\arctan\frac{1}{2}\right)+\left(\arctan\frac{1}{2}-\arctan\frac{1}{3}\right)+\left(\arctan\frac{1}{3}-\arctan\frac{1}{4}\right)+\cdots$$

$$S_N = \arctan 1 - \arctan\left(\frac{1}{N+1}\right)$$

$$\lim_{N\to\infty} S_N = \arctan 1 - \arctan 0 = \pi/4$$

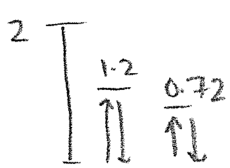
(c)

$$\left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{4}{5}\right)+\left(\frac{4}{5}-\frac{5}{6}\right)+\left(\frac{5}{6}-\frac{6}{7}\right)+\cdots = \sum\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right)$$

$$S_N = \frac{1}{2} - \frac{N+1}{N+2}$$

$$\lim_{N\to\infty} S_N = \frac{1}{2} - 1 = -\frac{1}{2}$$

2. $\boxed{13}$ A basketball is dropped from a height of 2m. It bounces back to a height of 1.2 m, then to a height of 0.72m, and continues bouncing back to a height 60% of its previous height on each bounce. Find the total vertical distance traveled as it bounces up and down.




$$\underset{\text{up}}{\sum_{n=0}^{\infty} 2(0.6)^n} + \underset{\text{down}}{\sum_{n=0}^{\infty} 1.2(0.6)^n}$$

$$= \frac{2}{1-0.6} + \frac{1.2}{1-0.6} = 5 + 3 = 8 \text{ m}$$

3. $\boxed{10}$ Evaluate $\displaystyle\int_2^\infty \frac{3x+2}{x^3+3x^2+2x}\,dx.$ HINT: $\displaystyle\frac{3x+2}{x^3+3x^2+2x}=\frac{1}{x}+\frac{1}{x+1}-\frac{2}{x+2}.$

$$= \lim_{R\to\infty}\int_2^R \frac{1}{x}+\frac{1}{x+1}-\frac{2}{x+2}\,dx = \lim_{R\to\infty}\left[\ln x+\ln(x+1)-2\ln(x+2)\right]$$

$$= \lim_{R\to\infty}\left[\ln\left(\frac{x^2+x}{(x+2)^2}\right)\right]\Bigg|_2^R = \lim_{R\to\infty}\ln\left(\frac{R^2+R}{(R+2)^2}\right)-\ln\left(\frac{6}{16}\right) = -\ln\left(\frac{6}{16}\right)$$

4. $\boxed{15}$ Consider the integral

$$\int_2^\infty \frac{\sqrt{x}+1}{x^2-1}\,dx. \qquad (1)$$

(a) Show that $f(x)=\dfrac{\sqrt{x}+1}{x^2-1}$ is decreasing for $x>2$.

HINT: $x^2-1=(x+1)(\sqrt{x}-1)(\sqrt{x}+1)$.

$$f(x)=\frac{1}{(\sqrt{x}-1)(x+1)}$$

$$f'(x)=\frac{-\left[\sqrt{x}-1+\frac{1}{2\sqrt{x}}(x+1)\right]}{[(x+1)(\sqrt{x}-1)]^2}<0$$

(b) Use an appropriate comparison to show $\displaystyle\sum_{n=2}^\infty \frac{\sqrt{n}+1}{n^2-1}$ converges.

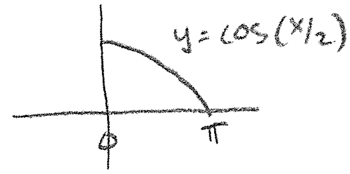
Direct Comp. : $0\le \dfrac{\sqrt{n}}{n^2}=\dfrac{1}{n^{3/2}}\le\dfrac{\sqrt{n}+1}{n^2-1}$ inconclusive

LCT $$\lim_{n\to\infty}\frac{\frac{\sqrt{n}+1}{n^2-1}}{\frac{1}{n^{3/2}}}=\lim_{n\to\infty}\frac{n^2+n^{3/2}}{n^2-1}=1>0$$

Because $\displaystyle\sum_{n=2}^\infty\frac{1}{n^{3/2}}$ converges $(p=3/2>1)$, then $\displaystyle\sum_{n=2}^\infty\frac{\sqrt{n}+1}{n^2-1}$ converges by LCT

(c) Use (a) and (b) to show (1) converges.

Because $f(x)=\dfrac{\sqrt{x}+1}{x^2-1}$ is positive, continuous, and decreasing on $[2,\infty)$, and $\displaystyle\sum_{n=2}^\infty\frac{\sqrt{n}+1}{n^2-1}$ converges, $\displaystyle\int_2^\infty\frac{\sqrt{x}+1}{x^2-1}\,dx$ must converge by Integral Test.




5. Consider the the graph of $y = \cos(x/2)$ for $x \in [0, \pi]$.

$$y' = \frac{-\sin(x/2)}{2}$$

$$(y')^2 = \frac{\sin^2(x/2)}{4}$$

(a) $\boxed{8}$ Express the arc length as an integral. Do not evaluate the integral.

$$S = \int_0^{\pi} \sqrt{1 + \tfrac{1}{4}\sin^2(x/2)}\, dx$$

(b) $\boxed{5}$ A surface is generated by rotating the graph about the $x$-axis; express the surface area as an integral. Do not evaluate the integral.

$$SA = \int_0^{\pi} 2\pi\left(\cos(x/2)\right)\sqrt{1 + \tfrac{1}{4}\sin^2(x/2)}\, dx$$

(c) $\boxed{2}$ A surface is generated by rotating the graph about the $y$-axis; express the surface area as an integral. Do not evaluate the integral.

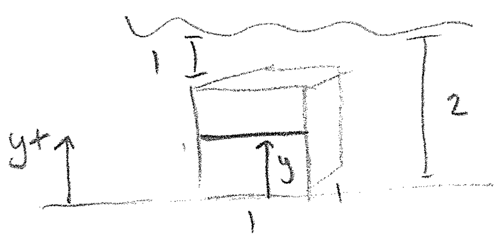
HINT: Integrate with respect with $x$.

$$SA = \int_0^{\pi} 2\pi x \sqrt{1 + \tfrac{1}{4}\sin^2(x/2)}\, dx$$

$$\left[\text{OR } SA = \int_0^1 2\pi\, 2\arccos y \sqrt{1 + \frac{4}{1-y^2}}\, dy\right]$$

6. $\boxed{10}$ A cube of side length 1 m is submerged in a fluid of density $\rho$. The top of the box is parallel to the surface of the fluid and 1 m below the surface. Find the fluid force on each of the six sides.

HINT: Four of the sides are the same.




top $F = 1 \cdot \rho g 1 = \rho g$ N

bottom $F = 1 \rho g 2 = 2\rho g$ N

all sides $F = 4\int_0^1 \rho g(2-y)\, dy$

$$= 4\rho g\left[2y - \frac{y^2}{2}\right]\Big|_0^1$$

$$= 4\rho g(3/2) = 6\rho g$$

7. $\boxed{10}$ Please indicate **True** or **F** false.

(a) **T** / **F**: The $\boxed{\text{sequence}}$ $\left\{\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \dots\right\}$ converges to $\frac{1/5}{1-1/5}$.

(b) **T** / **F** : The series $\sum_{n=0}^{\infty} \frac{4^{2n}}{5^{n+1}}$ diverges.

(c) **T** / **F**: If $a_n \to 0$ as $n \to \infty$, then $\sum a_n$ converges.

(d) **T** / **F** If $a_n \to 0$ as $n \to \infty$, then $\sum a_n$ diverges.

(e) **T** / **F**: If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\sum a_n$ converges.

8. $\boxed{9}$ For the following series, specify what series you would compare each to (using either the direct or limit comparison test), and then based on your comparison, decide if the series converges or diverges. No formal justification is required.

(a) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2+1}$ compare to $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ so it CONVERGES/DIVERGES.

(b) $\sum_{n=1}^{\infty} \frac{3}{n5^n}$ compare to $\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^n$ so it CONVERGES/DIVERGES.

(c) $\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^3+5n}}$ compare to $\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ so it CONVERGES/DIVERGES.

9. $\boxed{6}$ Consider the series given by

$$\sum_{n=0}^{\infty} x^n.$$

(a) For what values of $x$ does the series converge?

$|x| < 1$ (geometric)

(b) For those $x$, what does the series converge to?

$$\frac{1}{1-x}$$

| Page | 1 | 2 | 3 | 4 | Total |
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| Value | 25 | 25 | 25 | 25 | 100 |
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