Math 182 Final Exam **Name:** Key

1 May 2018 Point values in $\boxed{\text{boxes}}$.

1. Evaluate.

(a) $\boxed{5}$

$$\int \frac{\cos t}{\sin^2 t - 1}\, dt$$

or $\int \frac{\cos t}{-\cos^2 t}\, dt = \int -\sec t\, dt$

$u = \sin t$
$du = \cos t\, dt$

$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1) + B(u-1)$

let $u=1$: $1 = A2 \Rightarrow A = \frac{1}{2}$

let $u=-1$: $1 = B(-2) \Rightarrow B = -\frac{1}{2}$

$$= \int \frac{1}{u^2-1}\, du = \int \frac{A}{u-1} + \frac{B}{u+1}\, du$$

$$= \int \frac{1/2}{u-1} + \frac{-1/2}{u+1}\, du$$

$$= \tfrac{1}{2}\ln|\sin t - 1| - \tfrac{1}{2}\ln|\sin t + 1| + C$$

(b) $\boxed{5}$

$$\int \frac{\sin 2t}{1+\cos^2 t}\, dt = \int \frac{2\sin t\cos t}{1+\cos^2 t}\, dt$$

$u = \cos t$
$du = -\sin t\, dt$

$$= \int \frac{-2u}{1+u^2}\, du$$

$w = 1+u^2$
$dw = 2u\, du$

$$= \int -\frac{1}{w}\, dw = -\ln|1+\cos^2 t| + C$$

(c) $\boxed{5}$ WWCMD?[1]

$$\int \sin(t^2)\, dt = \int \sum_{n=0}^{\infty} \frac{(-1)^n (t^2)^{2n+1}}{(2n+1)!}\, dt$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+3}}{(4n+3)(2n+1)!} + A$$

[1]What would Colin Maclaurin do?

2. $\boxed{15}$ The largest of the Knights of the Elliptical Table in the Kingdom of Calculot was Sir Cumference[2]. If the Elliptical Table is given by the equation

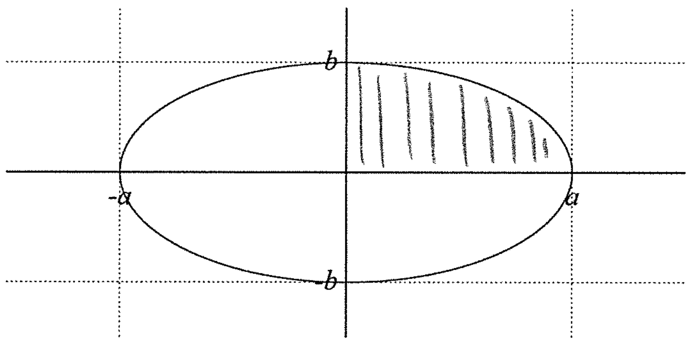
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

use trigonometric substitution to show that the area of the top of the table is

$$A = \pi ab.$$

[HINT 1:] Consider the top semi-ellipse given by $y = b\sqrt{1 - (x/a)^2}$.

[HINT 2:] The ellipse has the following graph.



$$A = 4\int_0^a b\sqrt{1-(x/a)^2}\, dx \qquad \frac{x}{a} = \sin\theta \qquad dx = a\cos\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta = 2ab\int_0^{\pi/2} 1 + \cos 2\theta\, d\theta$$

$$= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]\Bigg|_0^{\pi/2} = 2ab\left[\pi/2 + \underbrace{\frac{\sin\pi}{2}}_{0} - 0\right]$$

$$= \pi ab \checkmark$$

[2]He gained his largess by consuming too much pi!

3. Let

$$g(x) = \int_0^x \arcsin(2t)\, dt.$$

(a) $\boxed{10}$ Simplify $g(x)$, i.e. evaluate the integral.

IBP

$u = \arcsin(2t) \quad dv = dt$

$du = \frac{2}{\sqrt{1-(2t)^2}}\, dt \quad v = t$

$u = 1-4t^2$

$du = -8t\, dt$

$$g(x) = t\arcsin(2t)\Big|_0^x - \int_0^x \frac{2t}{\sqrt{1-4t^2}}\, dt$$

$$= (x\arcsin(2x) - 0) + \frac{1}{4}\int_1^{1-4x^2} \frac{1}{\sqrt{u}}\, du$$

$$= x\arcsin(2x) + \frac{1}{4}\cdot 2\sqrt{u}\Big|_1^{1-4x^2}$$

$$= x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2} - \frac{1}{2}$$

(b) $\boxed{8}$ Given that

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} \quad \text{for } x \in (-1,1),$$

find a power series representation for $g(x)$.

$$\arcsin(2t) = \sum_{n=0}^{\infty} \frac{(2n)!(2t)^{2n+1}}{(2^n n!)^2(2n+1)}$$

for $|2t| < 1$

$$g(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(2n)!\, 2^{2n+1}\, t^{2n+1}}{(2^n n!)^2(2n+1)}\, dt$$

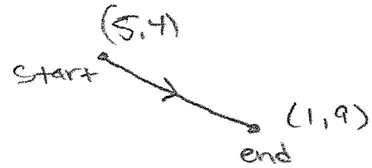
$$= \sum_{n=0}^{\infty} \frac{(2n)!\, 2^{2n+1}\, t^{2n+2}}{2^{2n}(n!)^2(2n+1)(2n+2)}\Bigg|_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n+2}}{(n!)^2(2n+1)(n+1)}$$

(c) $\boxed{2}$ For the series you found in (b), what is the radius of convergence?

$|2t| < 1 \quad \Rightarrow \quad R = 1/2$

4. $\boxed{5}$ Find parametric equations, $x(t)$ and $y(t)$, for the line segment starting at $(5,4)$ and ending at $(1,9)$. Include an appropriate domain for t.



$x = 5 + (1-5)t$

$y = 4 + (9-4)t$

$x = 5 - 4t$
$y = 4 + 5t$ for $0 \le t \le 1$

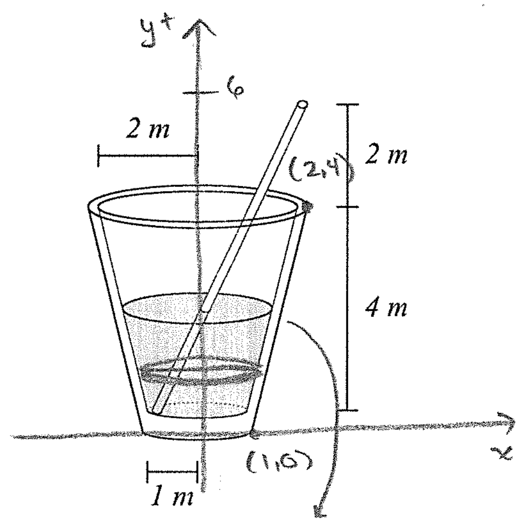
5. $\boxed{10}$ After a long climb to the top of a beanstalk of height 800m Jack sees a giant drinking a cloudy cocktail. The interior of the glass is in the shape of a frustum with lower radius 1 m and upper radius 2 m. The height of the interior of the glass is 4 m. The cocktail[3] she is drinking is very poorly mixed and has a density given by $\rho(y) = \kappa(12-y)$ where y measures the distance from the interior of the bottom of the glass. The giant is sipping her cocktail using a straw that extends 2 m past the top of the glass. When Jack arrived the giant had already consumed the 'top half' (as measured by height, not volume). How much work does the giant do by slurping the 'bottom half' up through the straw?

Express your solution as a definite integral; **do not evaluate the integral.** Please start by clearly identifying the coordinate system that you will be using.

$A_{slice} = \pi x^2 = \pi\left(\frac{y}{4}+1\right)^2$

$W_{slice} = \pi g \underbrace{\kappa(12-y)}_{\rho}\left(\frac{y}{4}+1\right)^2(6-y)\,\Delta y$

$$W = \int_0^2 \pi g \kappa (12-y)\left(\frac{y}{4}+1\right)^2(6-y)\,dy$$

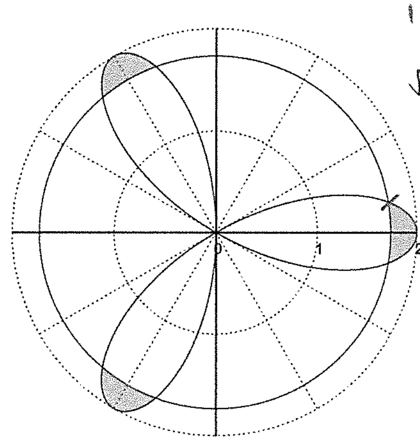


$y = 4(x-1)$

$\frac{y}{4} + 1 = x$

[3]Our very pleasant vegan giant appreciates a good margarita this time of the semester, and she surely deserves one.

6. $\boxed{15}$ Find the area inside $r = 2\cos 3\theta$ and outside $r = \sqrt{3}$, the shaded region in the figure.



intersect:

$\sqrt{3} = 2\cos 3\theta$

$\frac{\sqrt{3}}{2} = \cos 3\theta$

$\pi/6 = 3\theta$

$\pi/18 = \theta$

$$A = \frac{6}{2}\int_0^{\pi/18} (2\cos 3\theta)^2 - (\sqrt{3})^2 \, d\theta$$

$$= 3\int_0^{\pi/18} 4\cos^2 3\theta - 3 \, d\theta$$

$$= 3\int_0^{\pi/18} 2(1+\cos 6\theta) - 3 \, d\theta$$

$$= 3\int_0^{\pi/18} 2\cos 6\theta - 1 \, d\theta$$

$$= 3\left[\frac{\sin 6\theta}{3} - \theta\right]\Big|_0^{\pi/18} = 3\left[\frac{\sin \pi/3}{3} - \frac{\pi}{18}\right]$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

7. $\boxed{10}$ Find the length of the polar curve given by $r = \theta$ for $r \in [0, 2\pi]$.

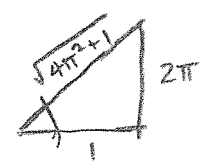
$$S = \int_0^{2\pi} \sqrt{\theta^2 + 1} \, d\theta$$

$\theta = \tan y$

$d\theta = \sec^2 y \, dy$

$$= \int_0^{\arctan 2\pi} \sqrt{\tan^2 y + 1} \sec^2 y \, dy = \int_0^{\arctan 2\pi} \sec^3 y \, dy$$

$$= \frac{1}{2}\left(\sec y \tan y + \ln|\sec y + \tan y|\right)\Big|_0^{\arctan 2\pi}$$



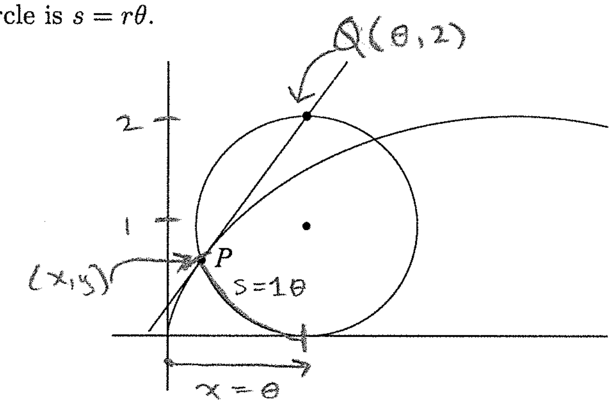
$$= \frac{1}{2}\left(\sqrt{4\pi^2+1}\cdot 2\pi + \ln\left|\sqrt{4\pi^2+1} + 2\pi\right| \underbrace{- 1\cdot 0 - \ln|1+0|}_{0}\right)$$

$$= \sqrt{4\pi^2+1}\cdot \pi + \frac{1}{2}\ln\left|\sqrt{4\pi^2+1} + 2\pi\right|$$

8. $\boxed{10}$ Show that the tangent line at a point P on the cycloid always passes through the top point on the rolling circle; see figure. Assume the generating circle has radius[4] 1 and the cycloid is parameterized by $c(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$.

[HINT 1:] θ parameterizes the angular displacement, i.e. the amount the circle has rolled[5].

[HINT 2:] The arc length of a sector of a circle is $s = r\theta$.



- $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{+\sin\theta}{1-\cos\theta}$$

- Slope of line through PQ:

$$m = \frac{2-y}{\theta - x} = \frac{2-(1-\cos\theta)}{\theta-(\theta-\sin\theta)} = \frac{1+\cos\theta}{+\sin\theta}$$

Show equal:

$$m = \frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)} = \frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta} \quad \checkmark$$

Thus, the line tangent to P must pass through the point at top of circle (b/c derivative at point P is the same as the slope of the line passing through both points).

Page	1	2	3	4	5	6	Total
Value	15	15	20	15	25	10	100
Points							

[4]You may recognize this as problem # 76 from the suggested exercises in 11.1. Although it would be a more interesting problem with an arbitrary radius ξ, we have decided to leave the question in its original form.

[5]See Hint 2.