1. Let \( g(x, y) = e^{x^2-y^2} \). Note that \( \nabla g(x, y) = \langle 2xe^{x^2-y^2}, -2ye^{x^2-y^2} \rangle \).

(a) (5 points) Find the maximum rate of change of \( g \) at \((1, 1)\).

\[
\nabla g(1, 1) = \langle 2, -3 \rangle \quad \text{so} \quad \max \ \text{P.R.C.} \quad || \nabla g(1, 1) || \approx \sqrt{13}
\]

(b) (5 points) Find the rate of change of \( g \) at \((1, 1)\) in the direction of \( \langle 2, 1 \rangle \).

\[
\nabla g(1, 1) \cdot \frac{\langle 2, 1 \rangle}{\sqrt{5}} = \frac{1}{\sqrt{5}}
\]

(c) (5 points) Find a direction in which the rate of change of \( g \) at \((1, 1)\) is zero.

We need a vector so that \( \nabla g(1, 1) \cdot \vec{v} = 0 \), one such vector is \( \langle 3, 2 \rangle \).

(d) (8 points) Find an equation for the tangent plane to the surface \( g(x, y) = e^{x^2-y^2} \) at the point \((1, 1)\).

\[
Z = g(1, 1) + g_x(1, 1)(x-1) + g_y(1, 1)(y-1)
\]

\[
= 1 + 2(x-1) - 3(y-1)
\]

(e) (4 points) Use the linear approximation to find an approximate value for \( z = g(x, y) \) when \( x = 1.1 \) and \( y = 0.9 \).

\[
g(1.1, 0.9) \approx 1 + 2(1.1-1) - 3(0.9-1) = 1.5
\]
2. (10 points) Calculate the partial derivative \( g_{zzwx} \) where \( g(x, y, z, w) = x^3 w^2 z^2 + \sin \left( \frac{xy}{z^2} \right) \).

By Clairaut's Theorem

\[
\frac{\partial^3 g}{\partial z^2 \partial w \partial x} = \frac{\partial}{\partial z} \left( \frac{\partial^2 g}{\partial w \partial x} \right)
\]

\[
\frac{\partial^2 g}{\partial w \partial x} = 2x^3 w^2 z
\]

\[
\frac{\partial^2 g}{\partial x \partial z^2} = 12x^2 w z
\]

\[
\frac{\partial^2 g}{\partial z^2 \partial w \partial x} = 12x^2 w
\]

3. (8 points) Given \( f(x, y) = (x^2 + y)e^{x-y} \), find the gradient of \( f \).

\[
\nabla f(x, y) = \left< f_x(x, y), f_y(x, y) \right>
\]

\[
= \left< 2xe^{x-y} + (x^2 + y)e^{x-y}, e^{x-y} - (x^2+y)e^{x-y} \right>
\]
4. (10 points) Let

\[ u(x, y, z) = \frac{xy^2}{z^3}, \quad x(p, r, t) = 3p + 6r^{16} + 5 \ln t, \quad y(p, r, t) = 5p - 6 \ln r + t^{\sin(t)}, \quad z(p, r, t) = 7p + 6r^4 - \frac{1}{t}. \]

Find \( \frac{\partial u}{\partial p} \)

\[
\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial p}
\]

\[
= \frac{y^2}{z^3} (3) + \frac{2xy}{z^3} (5) \Rightarrow 3 \frac{xy^2}{z^4} (7)
\]
5. (8 points total) Show that the limit does not exist.

\[
\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + 2y^2}
\]

Consider the limit in polar where \(2xy = 2(r \cos \theta)(r \sin \theta)\)

and \(x^2 + 2y^2 = x^2 + y^2 + y^2 = r^2 + r^2 \sin^2 \theta\)

So \(\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + 2y^2} = \lim_{r \to 0^+} \frac{2r^2 \cos \theta \sin \theta}{r^2 (1 + \sin^2 \theta)} = \lim_{r \to 0^+} \frac{2 \cos \theta \sin \theta}{1 + \sin^2 \theta}\)

Since the limit depends on \(\theta\), \(\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + 2y^2}\) does not exist.

6. (3 points) Two contour maps are shown. One is for a function \(f\) whose graph is a cone. The other is for a function \(g\) whose graph is a paraboloid. Clearly identify which contour map is that of a cone.

[Diagram of contour maps]

7. (4 points) Below is a level curve picture for the function \(f(x, y)\).

[Level curve picture]

Circle the best answer:

(a) \(f_x(2,1)\) equals i. \(-2\) ii. \(1/2\) iii. 1 iv. 2

(b) \(f_y(2,1)\) equals i. \(-1\) ii. 0 iii. \(1/2\) iv. 1
8. (15 points) Find all critical points of \( f(x, y) = x^2 + 3xy + y^2 - 4x - y \) and classify them (local maximum, local minimum, or saddle) using the Second Derivative Test.

\[ f_x = 2x + 3y - 4 \quad \Rightarrow \quad f_{xx} = 2 \]

\[ f_y = 3x + 2y - 1 \quad \Rightarrow \quad f_{yy} = 2 \]

\[ f_{xy} = f_{yx} = 3 \]

\( f_x = 0 \) when \( 2x + 3y = 4 \) \((1)\)

\( f_y = 0 \) when \( 3x + 2y = 1 \) \((2)\)

Multiplying \((1)\) by 3 and subtracting \((2)\) times 2 gives:

\[ 6x + 9y = 12 \]
\[ 6x + 4y = 2 \]

\[ 5y = 10 \quad \Rightarrow \quad y = 2 \]
\[ x = -1 \]

\((-1, 2)\) is the only critical point.

Since \( D = f_{xx} f_{yy} - (f_{xy})^2 = 4 - 9 = -5 < 0 \),

it is a saddle point.
9. (15 points) Find the maximum and minimum of \( f(x, y) = xy \) on the ellipse \( x^2 + 2y^2 = 8 \).

Using the method of Lagrange we have

\[
\nabla f = \lambda \nabla g \quad \text{where} \quad g(x, y) = x^2 + 2y^2 = 8 \quad (1)
\]

\[
y = \lambda (2x) \quad (2)
\]

\[
x = \lambda (4y) \quad (3)
\]

Solving (2) for \( \lambda \) gives \( \lambda = \frac{y}{2x} \)

(\( \text{or} \ x = y = 0 \) which is not a valid point on the ellipse)

Solving (3) for \( \lambda \) gives \( \lambda = \frac{x}{4y} \) (\( \text{or} \ x^2 = 4y \) is above)

Equating gives \( \frac{y}{2x} = \frac{x}{4y} \) so \( 2y^2 = x^2 \)

Substituting into (1) gives \( x^2 + x^2 = 8 \) \( \text{or} \ x^2 = 4 \) \( \text{or} \ x = \pm 2 \)

\[ f(2, \sqrt{2}) = 2\sqrt{2} \]
\[ f(-2, \sqrt{2}) = -2\sqrt{2} \]
\[
\begin{align*}
\text{Max} & \quad f(2, \sqrt{2}) = 2\sqrt{2} \\
\text{Min} & \quad f(-2, \sqrt{2}) = -2\sqrt{2}
\end{align*}
\]

\[ f(2, -\sqrt{2}) = -2\sqrt{2} \]
\[ f(-2, -\sqrt{2}) = 2\sqrt{2} \]