M273Q, Old Final A Name: SOLUTIONS

Instructions: Closed book. No calculator allowed. Show all work and use correct notation to receive full credit! Write legibly.

1. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ and f a scalar function of three variables. For each of the following, state whether the operations shown produce a vector field, a scalar function, or whether they cannot be computed, in which case the statement is nonsense.

   (a) **Vector** **Scalar** **Nonsense** $\nabla \cdot \mathbf{F}$

   (b) **Vector** **Scalar** **Nonsense** $\nabla(\nabla \cdot \mathbf{F})$

   (c) **Vector** **Scalar** **Nonsense** $\nabla \times (\nabla \times \mathbf{F})$

   (d) **Vector** **Scalar** **Nonsense** $(\nabla f) \times \mathbf{F}$

   (e) **Vector** **Scalar** **Nonsense** $(\nabla \cdot \mathbf{F}) \cdot \mathbf{F}$

2. Given the vectors $\mathbf{a} = <2, 1, 0>, \mathbf{b} = <2, -1, 2>$, and $\mathbf{c} = <0, 2, 1>$, find:

   (a) A vector of length 7 that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

   $$\begin{array}{r} <2,1,0> \\ \times <2,-1,2> \\ \hline <2,-4,-4> \end{array}$$

   $$\|<2,-4,-4\| = \sqrt{4+16+16} = \sqrt{36} = 6$$

   $$\frac{7}{6} < 2,-4,-4> = < \frac{14}{6}, \frac{-28}{6}, \frac{-28}{6}>$$

   (b) An equation for the plane that is parallel to both $\mathbf{a}$ and $\mathbf{b}$ and that goes through the point $(-1, 1, 2)$.

   $$1(x+1) - 2(y-1) - 2(z-2) = 0.$$

   or $$x - 2y - 2z = -7$$

   (c) The volume of the parallelopiped spanned by $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$.

   $$\text{Volume} = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = |<2,-4,-4> \cdot <0,2,1>|$$
   $$= 12$$

   (d) The cosine of the angle between $\mathbf{a}$ and $\mathbf{c}$.

   $$\vec{a} \cdot \vec{c} = \|\vec{a}\| \|c\| \cos\theta$$

   $$\cos\theta = \frac{\vec{a} \cdot \vec{c}}{\|\vec{a}\| \|\vec{c}\|} = \frac{2}{\sqrt{5} \cdot \sqrt{5}} = \frac{2}{5}$$

3. Let $\mathbf{r}(t) = < t^2, -2t, t >$ and $f(x, y, z) = x^2(y + z)$. $= x^2y + x^2z$

(a) At the point $(1, -2, 1)$ in what direction does f increase most rapidly?

$$\nabla f = < 2x(y+z), x^2, x^2 >$$

$$\nabla f(1,-2,1) = <-2, 1, 1> \longleftarrow \text{ in this direction.}$$

(b) Find the rate of change of f in the direction tangent to the curve $\mathbf{r}(t)$ at the point $(1, -2, 1)$.

$$\vec{r}'(t) = <2t, -2, 1> \qquad \vec{r}'(1) = <2, -2, 1>$$

rate of change $= D_{\frac{<2,-2,1>}{3}} f(1,-2,1)$ $\uparrow$ appropriate t since $\vec{r}(1) = <1, -2, 1>$.

$$= \frac{<-2, 1, 1> \cdot <2, -2, 1>}{3} = -5/3$$

(c) Use the chain rule to calculate $\frac{d}{dt}(f(\mathbf{r}(t))$ at $t = 1$.

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f(\vec{r}(1)) \cdot r'(1) = <-2, 1, 1> \cdot <2, -2, 1>$$

$$= -5.$$

4. Given that $\mathbf{r}(t) = < \cos 3t, \sin 3t, 4t >$ in position at time t, find:

(a) the velocity at time $t = \pi$.

$$\vec{r}'(t) = <-3\sin 3t, 3\cos 3t, 4>$$

$$\vec{r}'(\pi) = <0, -3, 4>$$

(b) the speed at time $t = \pi$.

$$\text{speed} = \|\vec{r}'(\pi)\| = 5$$

(c) the acceleration at time $t = \pi$.

$$\vec{r}''(t) = <-9\cos 3t, -9\sin 3t, 0>$$

$$\vec{r}''(\pi) = <9, 0, 0>$$

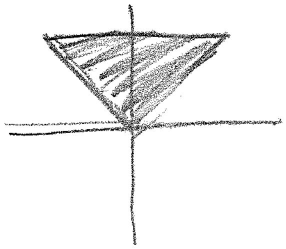
(d) the length of the path of motion at time $t = \pi$.

$$\text{length} = \int_0^{\pi} \|r'(t)\|\,dt = \int_0^{\pi} 5\,dt = 5\pi$$

(e) an equation for the tangent line to $\mathbf{r}(t)$ at the point $(-1, 0, 4\pi)$.

$$x = -1 \qquad y = -3t \qquad z = 4\pi + 4t.$$

5. Compute $\iint_{\mathcal{D}} x + y\, dA$ where $\mathcal{D}$ is the triangular domain with vertices $(-1,1),(1,1),(0,0)$.



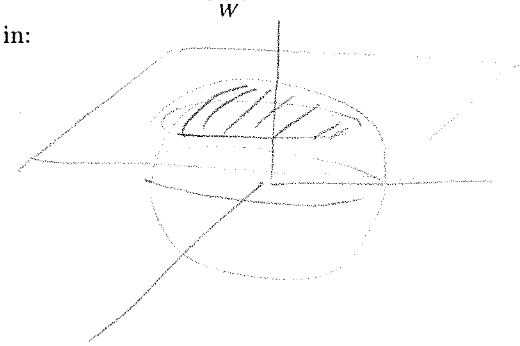
$$\int_0^1\int_{-y}^{y}(x+y)\,dx\,dy = \int_0^1 \frac{1}{2}x^2+xy\Big|_{x=-y}^{x=y} dy$$

$$= \int_0^1 2y^2\,dy = \frac{2}{3}y^2\Big|_0^1 = \frac{2}{3}$$

6. Set up but <u>do not evaluate</u>, iterated triple integrals with the appropriate limits for $\iiint_W x^2 + y^2\, dV$

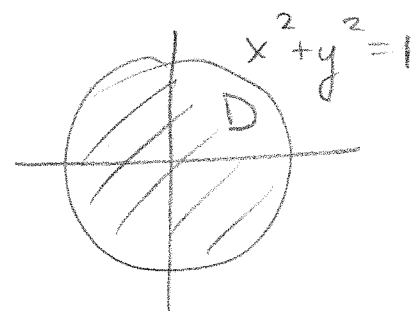
where W is the solid lying inside $x^2+y^2+z^2=2$ and above $z=1$, in:

(a) rectangular coordinates



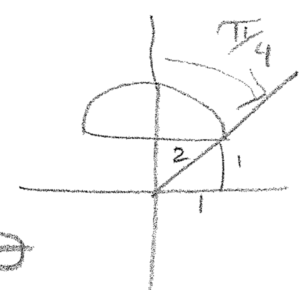
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1}^{\sqrt{2-x^2-y^2}} x^2+y^2\,dz\,dy\,dx$$

(b) cylindrical coordinates



$$\int_0^{2\pi}\int_0^1\int_1^{\sqrt{2-r^2}} r^3\,dz\,dr\,d\theta$$

(c) spherical coordinates



$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{\sqrt{2}} \rho^2\sin^2\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$\rho\cos\phi = z = 1$

$\rho = \frac{1}{\cos\phi} = \sec\phi$

7. Compute $\int\limits_C xz\,ds$, C is the straight line segment from $(1,2,3)$ to $(3,1,1)$.

$x = 1+2t \quad 0 \le t \le 1$

$y = 2-t$

$z = 3-2t$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

$= \sqrt{2^2+(-1)^2+(-2)^2}\,dt$

$= 3\,dt$

$= \int_0^1 (1+2t)(3-2t)\,3\,dt$

$= 3\int_0^1 3+4t-4t^2\,dt$

$= 11$

8. (a) Is $\mathbf{F}(x,y,z) = < 2x\cos y, \cos y - x^2\sin y, z >$ conservative? (justify).

$\nabla \times F = \langle 0, 0, -2x\sin y + 2x\sin y \rangle = \vec{0}$

so yes (all deriv's continuous)

(b) Calculate $\int\limits_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve $\mathbf{r}(t) = < te^t, t\pi, (1+t)^2 >$, $0 \le t \le 1$.

if $f(x,y,z) = x^2\cos y + \sin y + \frac{1}{2}z^2$ then

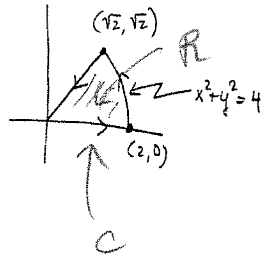
$\nabla f = \vec{F}$. So

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$= f(e,\pi,4) - f(0,0,1)$

$= \frac{15}{2} - e^2$

9. Use Green's Theorem to calculate $\int_C (x - y^3)dx + (x^3 - y)dy$, where C is the closed curve bounding the wedge shaped region pictured.



$$= \iint_R \frac{\partial}{\partial x}(x^3 - y) - \frac{\partial}{\partial y}(x - y^3)\,dA$$

$$= \iint_R 3x^2 + 3y^2\,dA$$

$$= \int_0^{\pi/4}\int_0^2 3r^2 r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{3}{4}r^4\Big|_0^2\,d\theta = \frac{\pi}{4}(12) = 3\pi.$$

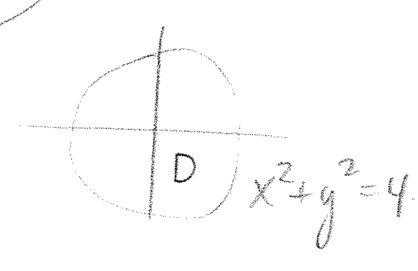
10. Find the surface area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$.

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dx\,dy$$

$$= \sqrt{(-2x)^2 + (-2y)^2 + 1}\;dx\,dy = \sqrt{4x^2 + 4y^2 + 1}\;dx\,dy$$

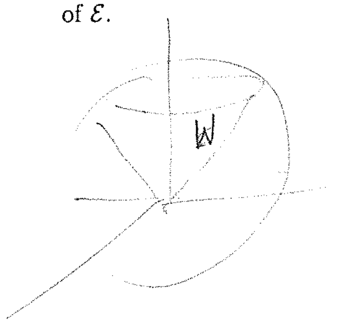
$$\Rightarrow \text{Surface Area} = \iint_S 1\,dS = \iint_D \sqrt{4x^2 + 4y^2 + 1}\;dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\;r\,dr\,d\theta$$



$$= \int_0^{2\pi} \frac{1}{8}\left(\frac{2}{3}\right)(1 + 4r^2)^{3/2}\Big|_0^2\,d\theta = \left(\frac{17^{3/2} - 1}{12}\right)(2\pi)$$

11. Let E be the solid region that is bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by $z = \sqrt{9-x^2-y^2}$. Calculate the flux of $\mathbf{F}(x,y,z) = < xy^2, yx^2, \frac{1}{3}z^3 >$ outwards across the boundary surface of $\mathcal{E}$.



$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint_W \operatorname{div} \vec{F} \, dV$$

$$= \iiint_W (y^2 + x^2 + z^2) \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{5}\rho^5 \sin\phi \Big|_0^3 \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{243}{5} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{243}{5}\right)(-\cos\phi)\Big|_0^{\pi/4} = 2\pi\left(\frac{243}{5}\right)\left(1 - \frac{\sqrt{2}}{2}\right).$$