M273Q, Old Final B Name: SOLUTIONS

Instructions: Closed book. No calculator allowed. Show all work and use correct notation to receive full credit! Write legibly.

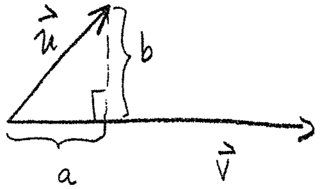
1. True or False? Circle ONE answer for each.

   True or [✓ False]: If **F** is a vector field, then div **F** is a vector field.

   True or [✓ False]: If **F** and **G** are vector fields, then curl $(\mathbf{F} \cdot \mathbf{G}) = \text{curl } \mathbf{F} \cdot \text{curl } \mathbf{G}$.

   [✓ True] or False: If S is a sphere and **F** is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

2. Given the vectors $\mathbf{v} = < 2, 1, -2 >, \mathbf{u} = < 3, 2, 1 >$, find the lengths of a and b pictured below:



$$a = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} = \frac{6+2-2}{3} = 2$$

$$b = \sqrt{\|\vec{u}\|^2 - a^2} = \sqrt{14-4} = \sqrt{10}.$$

3. (a) Find an equation for the line through the points $(1, 2, 3)$ and $(-1, 5, 4)$.

$x(t) = 1 - 2t$
$y(t) = 2 + 3t$
$z(t) = 3 + t$

(b) Find an equation for the plane that is perpendicular to the line in part 3a and passes through the point $(4, 0, 1)$.

$$-2(x-4) + 3(y-0) + 1(z-1) = 0$$

or $-2x + 3y + z = -7$

(c) At what point do the line in part 3a and the plane in part 3b intersect?

$$-2(1-2t) + 3(2+3t) + (3+t) = -7$$

$-t = 1 \qquad t = -1.$

$x = 3, \; y = -1 \quad z = 2 \qquad (3, -1, 2)$

4. Given $f(x, y, z) = \dfrac{x}{1 + xyz}$:

(a) At the point $(1, 0, 2)$ in what direction does f increase most rapidly?

$$\frac{\partial F}{\partial x} = \frac{(1+xyz) - x(yz)}{(1+xyz)^2} \qquad \frac{\partial f}{\partial x}(1,0,2) = 1$$

$$\frac{\partial f}{\partial y} = -x^2z(1+xyz)^{-2} \qquad \frac{\partial f}{\partial y}(1,0,2) = -2 \qquad \nabla f(1,0,2) = \langle 1,-2,0$$

$$\frac{\partial f}{\partial z} = -x^2y(1+xyz)^{-2} \qquad \frac{\partial f}{\partial z}(1,0,2) = 0$$

(b) At the point $(1, 0, 2)$ what is the rate of change of f in the direction of $< 3, -4, 0 >$?

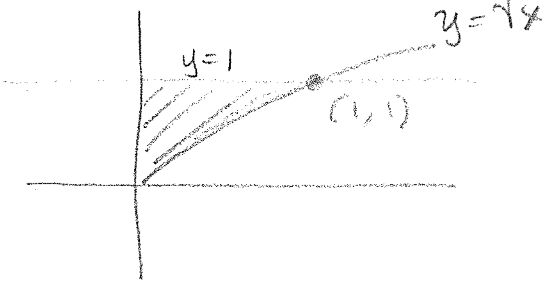
$$D_{\langle 3,-4,0\rangle} f(1,0,2) = \frac{\langle 1,-2,0\rangle \cdot \langle 3,-4,0\rangle}{5} = \frac{11}{5}.$$

5. Given $x = r\cos\theta$ and that r and θ depend on t in such a way that when $t = 0 : r = 2, \theta = \pi/4, \frac{dr}{dt} = 3, \frac{d\theta}{dt} = \pi$, find $\frac{dx}{dt}$ at $t = 0$.

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} = (\cos\theta)\frac{dr}{dt} - (r\sin\theta)\frac{d\theta}{dt}$$

$$= \left(\frac{3}{2} - \pi\right)\frac{\sqrt{2}}{2}.$$

6. Calculate $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1 + y^3}\, dy\, dx$.



$$= \int_0^1 \int_0^{y^2} (1+y^3)^{1/2}\, dx\, dy$$

$$= \int_0^1 y^2 (1+y^3)^{1/2}\, dy$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)(1+y^3)^{3/2}\Big|_0^1$$

$$= \frac{2}{9}\left(2^{3/2} - 1\right)$$

7. Let W be the region above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$. Calculate $\iiint\limits_W x^2 + y^2 \, dV$.

$$\iiint\limits_W x^2 + y^2 dV = \int_0^{2\pi}\int_0^1\int_r^1 r^3 \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1-r) r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4} r^4 - \frac{1}{5} r^5\right)\Big|_0^1 d\theta = \frac{1}{20}(2\pi) = \frac{\pi}{10}.$$

8. Verify that the vector field $\mathbf{F}(x, y, z) = < 2xy + z, x^2 + 1, x + 2z >$ is conservative and calculate the work done by $\mathbf{F}$ in moving an object from $(2, -1, 1)$ to $(1, 1, 0)$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+z & x^2+1 & x+2z \end{vmatrix} = \langle 0, 1-1, 2x-2x \rangle$$

$$= \vec{0}, \text{ conservative.}$$

$$f(x, y, z) = x^2 y + xz + y + z^2$$

$$\text{work} = f(1,1,0) - f(2,-1,1) = 2 - (-4 + 2 - 1 + 1)$$

$$= 4.$$

9. Calculate $\oint\limits_C -y^2\,dx + xy\,dy$ where C is the counterclockwise oriented simple closed curve consisting of the piece of the parabola $y = 1 - x^2$ between $(-1, 0)$ and $(1, 0)$ together with the piece of the x−axis between $(-1, 0)$ and $(1, 0)$.

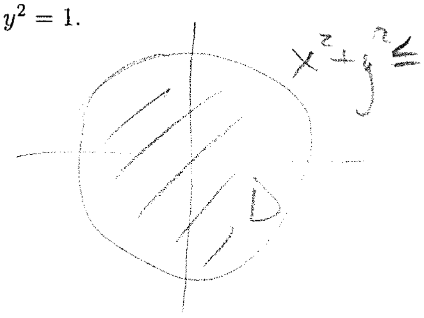
greens theorem

$$\oint_C -y^2dx + xy\,dy = \iint_D y + 2y\,dA = \int_{-1}^{1}\int_{0}^{1-x^2} 3y\,dy\,dx$$

$$= \int_{-1}^{1} \frac{3}{2}y^2\Big|_{y=0}^{y=1-x^2} dx = \frac{3}{2}\left(\int_{-1}^{1} 1 - 2x^2 + x^4\,dx\right)$$

$$= \frac{3}{2}\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_{-1}^{1} = \frac{8}{5}.$$

10. Find the surface area of the part of surface $z = xy$ that lies <u>within</u> the cylinder $x^2 + y^2 = 1$.



$$SA = \iint_D \sqrt{1+y^2+x^2}\,dA$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

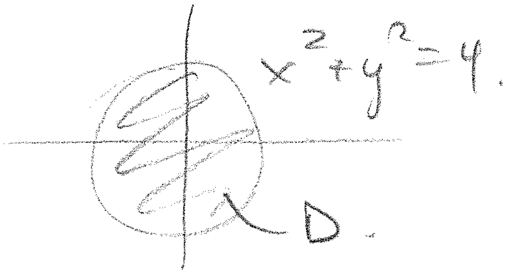
$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\;r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(r^2+1)^{3/2}\Big|_0^1 d\theta = \frac{2\pi}{3}(2\sqrt{2}-1).$$

11. Let $\mathcal{S}$ be the part of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$, oriented with upwards pointing normal vector, and let $\mathbf{F}(x, y, z) =< -y, x, z >$. Using Stokes' Theorem, calculate $\iint\limits_{\mathcal{S}} (curl(\mathbf{F})) \cdot d\mathbf{S}$.

Stokes: $\iint\limits_S \text{curl}\, \vec{F} \cdot d\vec{S} = \oint\limits_C \vec{F} \cdot d\vec{r}$

$$= \oint\limits_C -y\,dx + x\,dy + \overset{0}{\cancel{z\,dz}}$$



$$= \oint\limits_C -y\,dx + x\,dy$$

$$\overset{\text{greens}}{=} \iint\limits_D (1+1)\,dA$$

$$= 2\,\text{Area}(D) = 2\pi(2^2) = 8\pi.$$

12. Let $\mathbf{F}(x, y, z) =< x, y, z^2 >$ and let $\mathcal{S}$ be the closed surface consisting of the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq \sqrt{2}$, and the spherical cap $z = \sqrt{4 - x^2 - y^2}$, $\sqrt{2} \leq z \leq 2$. Using the divergence theorem, calculate the flux, $\iint\limits_{\mathcal{S}} \mathbf{F} \cdot d\mathcal{S}$, of $\mathbf{F}$ outwards across $\mathcal{S}$.

divergence thm.

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iiint\limits_W \text{div}\, \vec{F}\, dV = \iiint\limits_W 2z\, dV.$$

$$\overset{\text{spherical}}{=} \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 2\rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\int_0^{2\pi} \int_0^{\pi/4} \left(\frac{1}{4}\rho^4 \Big|_0^2\right) \cos\phi \sin\phi\, d\phi\, d\theta = \frac{1}{2}(2^4)\left(-\frac{1}{2}\cos^2\phi\right)\Big|_0^{\pi/} \cdot 2\pi$$

$$= 4\pi$$