Today we are interested in the flux through a silo $S$. For the purpose of this worksheet, define a silo to be the cylinder $x^2 + y^2 = 4$ for $0 \leq z \leq 6$ with a hemispherical cap on the top. There is no bottom on our silo. Normal vectors point outward.

Figure 1: My grandpa's farm, complete with silos.

On this page we are interested in using The Divergence Theorem to compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = (x + \cos(y^2 + z^{2016}), y + \sin(x^2 + z^{2016}), y^2)$.

1. Since the Divergence Theorem requires the surface be closed, we start need to do a little work up front.

   (a) Let $S_f$ be the surface representing the floor with downward normal. Compute the surface integral

   $$\mathbf{\vec{N}} = \langle 0, 0, -1 \rangle \quad \iint_{S_f} \mathbf{F} \cdot d\mathbf{S}.$$  

   $$\mathbf{\vec{F}} = \langle ?, ?, r^2 \sin^2 \theta \rangle$$  

   $$\int_0^{2\pi} \int_0^2 r \cdot 2 \int_0^{\pi} \left( \vec{\nabla} \cdot \mathbf{F} \right) \cdot \mathbf{dS} = \frac{1}{2} \int_0^{2\pi} \int_0^2 \left( \vec{\nabla} \cdot \mathbf{F} \right) \cdot \mathbf{dS} = \int_0^{2\pi} (-\frac{3}{4}) \int_2^r = \pi \left( -4 \right) = -4\pi$$

(b) Since $S + S_f$ is now a closed surface, use the Divergence Theorem to compute

   $$\iint_{S+S_f} \mathbf{F} \cdot d\mathbf{S} = 2 \left( \pi (4)(4) + \frac{1}{2} \cdot \frac{4}{3} \pi (2)^3 \right) = 48\pi + \frac{32\pi}{3} = \frac{17\pi}{3}$$

(c) Subtract to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{17\pi}{3} - (-4\pi) = \frac{18\pi}{3}$$
On this page, we are interested in using Stokes' Theorem in two different ways to compute

\[ \int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S}. \]

Below are two vector field \( \mathbf{F} \). For each, choose an appropriate way to use Stokes' Theorem and then do so to evaluate the integral.

2. \( \mathbf{F} = \langle xz^{2016} - y, x + y + z \sin (x^{2016}), e^{x^2+y^2} + z^{2016} \rangle \)

We could squash the silo into a disk, or use Stokes' to evaluate a line integral; we do it the 2nd way.

\[ \mathbf{\overrightarrow{r}} = \langle 2 \sin \theta, 2 \cos \theta, 0 \rangle, \quad \mathbf{\overrightarrow{r}'} = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle \]

\[ \mathbf{F}(\mathbf{r}) = \langle -2 \sin \theta, 2 \cos \theta + 2 \sin \theta, e^r \rangle \]

\[ \mathbf{F} \cdot \mathbf{\overrightarrow{r}'} = 4 \sin^2 \theta + 4 \cos^2 \theta + 4 \cos \theta \sin \theta = 4 + 4 \cos \theta \sin \theta \]

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} (4 + 4 \cos \theta \sin \theta) \, d\theta = 4 \theta + 2 \sin^2 \theta \bigg|_0^{2\pi} = 8\pi \]

3. \( \mathbf{F} = \langle yz^2 + xz^{2016}, xy^2 + y^{4.26} + ze^x, x^2y^2 - xyz \rangle \)

\[ \text{curl} (\mathbf{F}) = \langle -3z^2, 3y^2, y^2 - z^2 \rangle \quad \mathbf{N} = \langle 0, 0, r \rangle \quad \text{for the disk} \]

\[ \text{curl} (\mathbf{F}) \cdot \mathbf{N} = y^2 r = r^3 \sin^2 \theta \quad \text{using the std. param. of the disk, i.e.} \quad \langle r \cos \theta, r \sin \theta, 0 \rangle \]

\[ \int \int_S \text{curl} (\mathbf{F}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 r^3 \sin^2 \theta \, dr \, d\theta = 4\pi \]

See front.