Math 273 Quiz 4A Name: ________________

5 Feb 2016 Show appropriate work. Point Values in $\boxed{\text{boxes}}$.

Yesterday we defined the curvature as the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length, i.e.

$$\kappa(t) = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

However, using the definition is typically hard in that it requires an arc length parameterization to be explicity found for the curve. We would like to consider two alternative forms for the curvature. An application of the chain rule gives that

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

Since $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$, a little algebra allows us to write

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}. \tag{1}$$

$\boxed{5}$ Using (1) above, compute the curvature of $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$ for an arbitrary t and for $t = 1$.

$$\vec{r}\,'(t) = \langle e^t(\cos t - \sin t),\ e^t(\sin t + \cos t) \rangle$$

$$\|\vec{r}\,'(t)\| = e^t\left(\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t\right)^{1/2}$$

$$= \sqrt{2}\,e^t$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{2}}\langle \cos t - \sin t,\ \sin t + \cos t \rangle$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\sin t - \cos t,\ \cos t - \sin t \rangle$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{2}}\left(\sin^2 t + 2\sin t\cos t + \cos^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t\right)^{1/2}$$

$$= \frac{1}{\sqrt{2}}(\sqrt{2}) = 1$$

$$\kappa(t) = \frac{1}{\sqrt{2}\,e^t} \quad \text{so} \quad \kappa(1) = \frac{1}{\sqrt{2}\,e}$$

Although we can use (1) on the previous page to compute the curvature, it can quickly get out of hand. In many cases, an additional alternative form is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{2}$$

$\boxed{5}$ Using (2) above, compute the curvature of $\mathbf{r}(t) = \langle 2t, t^2, t^3/3 \rangle$ for an arbitrary t and for $t = 1$.

$\vec{r}(t) = \langle 2t, t^2, t^3/3 \rangle$

$\vec{r}\,'(t) = \langle 2, 2t, t^2 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 2t \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix} = \langle 2t^2, -4t, 4 \rangle$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{4t^4 + 16t^2 + 16} = 2\sqrt{(t^2+2)^2} = 2(t^2+2)$$

$$\|\vec{r}\,'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2$$

$$\kappa(t) = \frac{2(t^2+2)}{(t^2+2)^3} = \frac{2}{(t^2+2)^2}$$

$$\kappa(1) = \frac{2}{9}$$