Math 274 Section:_____ Name:____________________

Exam 1 - Friday, 22 September 2017

Show all appropriate work to receive full credit. Point values are in $\boxed{\text{boxes}}$

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at Montana State University, and that I have neither given nor received any unauthorized assistance (calculator, cell phone, notes, other people's exams, etc.) on this exam. Furthermore, I have and will not discuss the content of this exam, or provide any written or verbal aid to others. I understand that if I violate this statement, I will receive an exam score of 0."

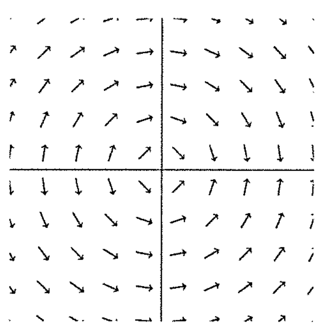
SIGNATURE:______________________________

1. $\boxed{6}$ Match each direction field below with the differential equation it corresponds to from the following list. Use the space below each to write the letter (A, B, or C) corresponding.

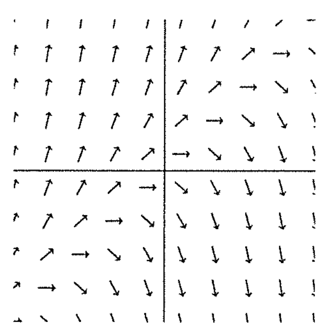
A. $\dfrac{dy}{dx} = x^2 - y^2$

B. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

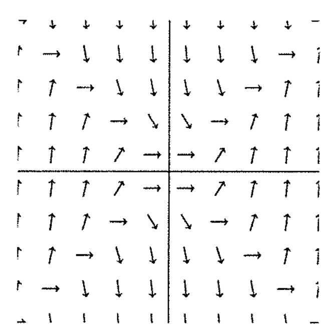
C. $\dfrac{dy}{dx} = y - x$



Equation B



Equation C



Equation A

2. $\boxed{6}$ Please indicate if the following statements are **T**rue or **F**alse.

(a) **T** / $\boxed{\textbf{F}}$: The initial value problem

$$xy' + y = xe^{x^{2017}}, \qquad y(9) = 22$$

has a unique solution $y(x)$ defined for all $x \in (-\infty, \infty)$.

Std Form is

$y' + \frac{1}{x} = e^{x^{2017}}$

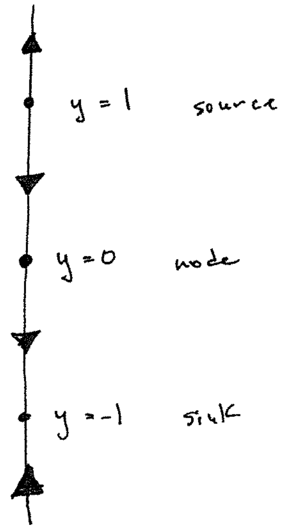
(b) $\boxed{\textbf{T}}$ / **F** : $\mathrm{Re}\left((1+i)e^{i\theta}\right) = \cos\theta - \sin\theta$.

(c) **T** / $\boxed{\textbf{F}}$: $y = 0$ is an equilibrium solution of $y' = y^2 - 1$.

3. Consider the autonomous equation

$$\frac{dy}{dt} = \frac{y^2(y^2-1)}{2}.$$

(a) $\boxed{6}$ Sketch the phase line for the equation and classify each equilibria as a sink, source, or node.



(b) $\boxed{2}$ Use the phase line to predict the asymptotic behavior as $t \to \infty$ of the solution satisfying $y(0) = 0.9$.

$$y \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty$$

(c) $\boxed{6}$ Solve the equation to find an implicit solution satisfying $y(1) = 2$.

HINT: It is useful to know: $\dfrac{2}{y^2(y^2-1)} = \dfrac{1}{y-1} - \dfrac{1}{y+1} - \dfrac{2}{y^2}$.

$$\int \frac{2}{y^2(y^2-1)}\,dy = \int dt \qquad \text{so} \int \left(\frac{1}{y-1} - \frac{1}{y+1} - \frac{2}{y^2} \right) dy = t + C$$

$$\ln|y-1| - \ln|y+1| + \frac{2}{y} = t + c$$

substituting $t=1$, $y=2$ gives

$$\ln 1 - \ln 3 + 1 = 1 + C \qquad \text{so} \quad C = -\ln 3$$

giving

$$\ln|y-1| - \ln|y+1| + \frac{2}{y} = t - \ln 3$$

4. Exact Equations.

(a) $\boxed{3}$ Show the equation

$$\left(\frac{2x}{y}+1\right)dx+\left(\frac{x+e^y}{y}\right)dy=0$$

is not exact.

$$\frac{\partial}{\partial y}\left(\frac{2x}{y}+1\right) = -\frac{2x}{y^2}$$

since $\frac{\partial}{\partial y}M \neq \frac{\partial}{\partial x}N$

$$\frac{\partial}{\partial x}\left(\frac{x+e^y}{y}\right) = \frac{1}{y}$$

the equation is not exact.

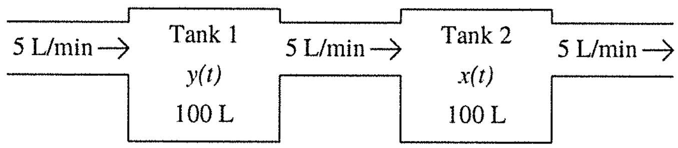
(b) $\boxed{5}$ Multiplying the above equation by y on both sides leads to the equation

$$(2x+y)dx+(x+e^y)dy=0.$$

This new equation is exact; find an implicit general solution.

$$x^2 + xy + e^y = C$$

5. Consider a two tank system as in the figure.



(a) Tank 1 is initially filled with 100 L of pure water. A brine solution containing 0.2 kg of salt per liter is being pumped into the tank at a rate of 5 L/min. The tank is well mixed and drains at 5 L/min. Let $y(t)$ be the amount of salt (in kg) in the tank at time t (in min).

i. $\boxed{4}$ Set up an initial value problem (a differential equation with an initial condition) modeling the amount of salt in Tank 1 at time t. **DO NOT SOLVE.**

$$\frac{dy}{dt} = (5)(0.2) - (5)\left(\frac{x}{100}\right), \quad y(0) = 0$$

ii. $\boxed{2}$ As $t \to \infty$, what can you say about the amount of salt in the Tank 1?

$$y \longrightarrow 20 \text{ Kg as } t \longrightarrow \infty$$

(b) Tank 2 is initially filled with 100 L of a brine solution with a concentration of 0.3 kg/L. The solution flowing out of Tank 1 flows into Tank 2 at 5 L/min. Tank 2 is also well stirred and drains at 5 L/min. Let $x(t)$ be the amount of salt (in kg) in Tank 2 at time t (in min).

i. $\boxed{10}$ The amount of salt in Tank 2 is modeled by the initial value problem

$$\frac{dx}{dt} = 1 - e^{-t/20} - \frac{x}{20}, \qquad x(0) = 30.$$

Solve the initial value problem.

$$\frac{dx}{dt} + \frac{x}{20} = 1 - e^{-t/20} \quad \text{is linear}$$

The integrating factor is $\mu(t) = e^{\int \frac{dt}{20}} = e^{t/20}$.

The solution is

$$x = e^{-t/20} \int \left(e^{t/20} - 1\right) dt = e^{-t/20}\left[20 e^{t/20} - t + C\right]$$

$$= 20 - e^{-t/20}(t + C)$$

$x(0) = 30$ so

$30 = 20 - C$ so $C = -10$ $\qquad x(t) = 20 + (10 - t)e^{-t/20}$

6. $\boxed{15}$ The differential equation

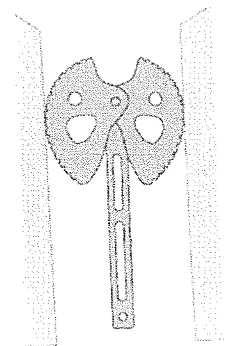
$$\frac{dy}{dx} = \frac{Cy - x}{Cx + y},$$

with C a positive constant, comes up in the design of the cams used in a spring-loaded camming device, i.e. a friend.

Use an appropriate substitution to find a general solution to the equation when $C = 1$, i.e. solve

$$\frac{dy}{dx} = \frac{y - x}{x + y}.$$

Express your solution as an implicit function in terms of x and y.



$$\frac{dy}{dx} = \frac{y-x}{x+y} = \frac{\frac{y}{x} - 1}{1 + \frac{y}{x}}$$

Let $v = \frac{y}{x}$ so $xv = y$

and $v + x\frac{dv}{dx} = \frac{dy}{dx}$

$$v + x\frac{dv}{dx} = \frac{v-1}{1+v}$$

$$x\frac{dv}{dx} = \frac{v-1}{1+v} - v = \frac{v-1}{1+v} - \frac{v+v^2}{1+v} = \frac{-1-v^2}{1+v} = -\frac{1+v^2}{1+v}$$

Separating yields

$$\int \frac{1+v}{1+v^2}\,dv = -\int \frac{dx}{x}$$

$$\arctan v + \frac{1}{2}\ln(1+v^2) = -\ln|x| + C$$

$$\arctan\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left(1 + \left(\frac{y}{x}\right)^2\right) = C - \ln|x|$$

7. Recently there have been a number of major hurricanes in the news. A simplified hurricane wind intensity model is given by

$$\frac{dU}{dt} = U(1 - U^n)$$

where U is the nondimensional wind speed and n is a positive constant that measures how close the wind intensity can come to the maximum potential intensity.

For the purpose of this exam we further simplify by letting $n = 2$ resulting in the equation

$$\frac{dU}{dt} = U(1 - U^2). \tag{1}$$

Verify that

$$U(t) = U_0 e^t \left(1 + U_0^2(e^{2t} - 1)\right)^{-1/2} \tag{2}$$

solves equation (1) with initial condition $U(0) = U_0$. In other words,

(a) $\boxed{2}$ verify (2) satisfies $U(0) = U_0$, and

$$U(0) = U_0 \left(1 + U_0^2 (1-1)\right)^{-1/2} = U_0 \quad ✓$$

(b) $\boxed{8}$ verify that (2) satisfies (1).

$$\frac{dU}{dt} = \underbrace{U_0 e^t \left(1 + U_0^2 (e^{2t} - 1)\right)^{-1/2}}_{U} + U_0 e^t \left(-\tfrac{1}{2}\right)\left(1 + U_0^2 (e^{2t} - 1)\right)^{-3/2} \left(2U_0^2 e^{2t}\right)$$

$$= U - \underbrace{U_0^3 e^{3t} \left(1 + U_0^2 \left(e^{2t} - 1\right)\right)^{-3/2}}_{U^3}$$

$$= U - U^3$$

$$= U(1 - U^2)$$

Problem	1	2	3	4	5	6	7	Total
Value	6	6	14	8	16	15	10	75
Points								