Math 274   Heaviside Thing

1. Express the following functions using step functions and determine their Laplace transforms.

(a) $f(t) = \begin{cases} 0, & t < 1 \\ 1, & 1 < t < 2 \\ 2, & 2 < t \end{cases}$ $= 0 + u(t-1)(1-0) + u(t-2)(2-1)$

$= u(t-1) + u(t-2)$

so $F(s) = \dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s}$

(b) $g(t) = \begin{cases} t, & t < \pi \\ e^{3t}\cos 2t, & \pi < t \end{cases}$ $= t + u(t-\pi)\left(e^{3t}\cos 2t - t\right)$

$G(s) = \dfrac{1}{s^2} + e^{-\pi s} \mathcal{L}\left\{ e^{3t+3\pi}\cos(2t+2\pi) - (t+\pi) \right\}$ Note: $\cos(2t+2\pi)$

$= \dfrac{1}{s^2} + e^{-\pi s}\left[e^{3\pi} \cdot \dfrac{s-3}{(s-3)^2 + 2^2} - \dfrac{1}{s^2} - \dfrac{\pi}{s} \right]$ $= \cos 2t$

2. Determine the inverse Laplace transform for the following.

(a) $H(s) = \dfrac{1}{s} + \dfrac{e^{-2s}}{s^2} + \dfrac{e^{-4s}}{s^3}$

$h(t) = 1 + u(t-2)(t-2) + \dfrac{1}{2} u(t-4)(t-4)^2$

(b) $K(s) = \dfrac{8e^{4-2s}}{(s-2)(s^2+4)}$ $= e^4 c^{-2s}\left[\dfrac{8}{(s-2)(s^2+4)} \right]$

$\dfrac{8}{(s-2)(s^2+4)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+4}$ so $k(s) = c^4 e^{-2s}\left[\dfrac{1}{s-2} - \dfrac{s}{s^2+4} - \dfrac{2}{s^2+4} \right]$

$8 = A(s^2+4) + (Bs+c)(s-2)$ so $k(t) = c^4 u(t-2)\left[e^{2(t-2)} - \cos(2t-4) - \sin(2t-4) \right]$

Let $s = 2 \Rightarrow A = 1$

Eq. Coeff

s^2: $0 = A + B \Rightarrow B = -1$

s^0: $8 = 4A - 2C \Rightarrow C = -2$

3. Applying the Laplace transform to the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, y'(0) = 0$$

with

$$f(t) = \begin{cases} 10e^t, & t < 1 \\ -8e^{2t}, & 1 < 2 < t \\ 0, & 2 < t \end{cases}$$

gives

$$Y(s) = \frac{10}{(s-1)(s^2+4)} + e^{-s}\left[\frac{-8e^2}{(s-2)(s^2+4)} - \frac{10e}{(s-1)(s^2+4)}\right] + e^{-2s}\left[\frac{8e^4}{(s-2)(s^2+4)}\right].$$

Determine $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, the solution to the given initial value problem.

You should reference the partial fraction decomposition in 2(b). Additionally, a routine exercise shows $\dfrac{10}{(s-1)(s^2+4)} = \dfrac{2}{s-1} - \dfrac{2s+2}{s^2+4}$. From 2(b) we know $\dfrac{8}{(s-2)(s^2+4)} = \dfrac{1}{s-2} - \dfrac{s+2}{s^2+4}$

So

$$Y(s) = \left[\frac{2}{s-1} - \frac{2s+2}{s^2+4}\right] + e^{-s}\left[-e^2\left(\frac{1}{s-2} - \frac{s+2}{s^2+4}\right) - e\left(\frac{2}{s-1} - \frac{2s+2}{s^2+4}\right)\right] + e^{-2s}e^4\left[\frac{1}{s-2} - \frac{s+2}{s^2+4}\right]$$

giving

$$y(t) = 2e^t - 2\cos 2t - \sin 2t$$
$$+ u(t-1)\left[-e^{+2}\left(e^{2(t-1)} - \cos(2t-2) - \sin(2t-2)\right)\right.$$
$$\left. -e\left(2e^{t-1} - 2\cos(2t-2) - \sin(2t-2)\right)\right]$$
$$+ u(t-2)\left[e^4\left(e^{2(t-2)} - \cos(2t-4) - \sin(2t-4)\right)\right]$$

Exercise: Using your favorite piece of technology, generate a picture of the graph of y. It should look something like this.